

STABILITY OF LINE SOLITONS FOR THE KP-II EQUATION IN \mathbb{R}^2

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ABSTRACT. We prove nonlinear stability of line soliton solutions of the KP-II equation with respect to transverse perturbations that are exponentially localized as $x \rightarrow \infty$. We find that the amplitude of the line soliton converges to that of the line soliton at initial time whereas jumps of the local phase shift of the crest propagate in a finite speed toward $y = \pm\infty$. The local amplitude and the phase shift of the crest of the line solitons are described by a system of 1D wave equations with diffraction terms.

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1. INTRODUCTION

The KP-II equation

$$(1.1) \quad \partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x, y) \in \mathbb{R}^2,$$

is a generalization to two spatial dimensions of the KdV equation

$$(1.2) \quad \partial_t u + \partial_x^3 u + 3\partial_x(u^2) = 0,$$

and has been derived as a model in the study of the transverse stability of solitary wave solutions to the KdV equation with respect to two dimensional perturbation when the surface tension is weak or absent. See [17] for the derivation of (1.1). Note that every solution of the KdV equation (1.2) is a planar solution of the KP-II equation (1.1).

The global well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ ($s \geq 0$) on the background of line solitons has been studied by Molinet, Saut and Tzvetkov [29] whose proof is based on the work of Bourgain [8]. For the other contributions on the Cauchy problem of the KP-II equation, see e.g. [12, 13, 15, 16, 35, 36, 37, 38] and the references therein.

Let

$$\varphi_c(x) = c \operatorname{sech}^2\left(\sqrt{\frac{c}{2}}x\right).$$

It is well known that the 1-soliton solution $\varphi_c(x - 2ct)$ of the KdV equation (1.2) is orbitally stable because φ_c is a minimizer of the Hamiltonian of (1.2) restricted on the manifold $\{u \in H^1(\mathbb{R}) \mid \|u\|_{L^2} = \|\varphi_c\|_{L^2}\}$. See [2, 4] and [5, 11, 40] for stability of solitary wave solutions of Hamiltonian systems. We remark that the KP-II equation does not fit into those standard argument because the first two terms of the Hamiltonian of the KP-II equation

$$\int (u_x^2(t, x, y) - 3(\partial_x^{-1} \partial_y u(t, x, y))^2 - 2u^3(t, x, y)) dx dy,$$

have the opposite sign. Recently, Mizumachi and Tzvetkov ([26]) have proved that $\varphi_c(x - 2ct)$ is orbitally stable as a solution of the KP-II equation in $L^2(\mathbb{R}_x \times \mathbb{T}_y)$. They used the Bäcklund transformation to prove that $L^2(\mathbb{R}_x \times \mathbb{T}_y)$ -stability follows from the L^2 -stability of the 0-solution, which is an immediate consequence of the conservation law of the $L^2(\mathbb{R}_x \times \mathbb{T}_y)$ -norm.

Unlike the perturbations which are periodic in the transverse directions, the perturbations in $L^2(\mathbb{R}^2)$ does not allow phase shifts of line solitons that are uniform in the transverse direction. This is because the difference of any translated line solitons and itself has infinite $L^2(\mathbb{R}^2)$ -mass whereas the well-posedness result [29] tells us the perturbation to the line soliton stay in $L^2(\mathbb{R}^2)$ for all the time. In order to analyze modulation of line solitons, we express solutions around the line soliton as

$$(1.3) \quad u(t, x, y) = \varphi_{c(t, y)}(z) - \psi_{c(t, y)}(x - x(t, y) + 4t) + v(t, x - x(t, y), y),$$

where $c(t, y)$ and $x(t, y)$ are the local amplitude and the local phase shift of the modulating line soliton, v is a remainder part which is expected to behave like an oscillating tail and $\psi_{c(t, y)}$ is an auxiliary function so that

$$(1.4) \quad \int_{\mathbb{R}} v(t, x, y) dx = \int_{\mathbb{R}} v(0, x, y) dx \quad \text{for any } t > 0,$$

Eq. (1.4) means that if the line soliton is locally amplified, then small waves are emitted from the rear of the line soliton. By introducing the auxiliary function $\psi_{c(t,y)}$, we have $v(t) \in L^2(\mathbb{R}^2)$ for every $t \geq 0$ and we are able to show that the L^2 -norm of v is almost conserved. We find that local modulations of the amplitude and phase shift can be described by a system of 1-dimensional wave equations with diffraction (viscous damping) terms, that a modulating line soliton converges to a line soliton with the same height as the original soliton on any compact subset of \mathbb{R}^2 (Theorem 1.1) and that “jumps” of the phase shift of the modulating line soliton propagate toward $y = \pm\infty$ along the crest of line solitons, which makes the set of all line soliton solutions unstable (Theorem 1.2).

Using geometric optics, Pedersen ([30]) heuristically explained that the amplitude and the orientation of the crest are described by a system of the Burgers equation. Since both the KP-II equation and the Boussinesq equation are long wave models for the 3D shallow water waves, it is natural to expect the same phenomena for KP-II. We find that the first order asymptotics of $\partial_y x(t, y)$ and $c(t, y)$ around $y = \pm(8c_0)^{1/2}t + O(\sqrt{t})$ are given by self-similar solutions of the Burgers equations as $t \rightarrow \infty$ (Theorem 1.3).

Now let us introduce our results. The first result is the stability of line soliton solutions for exponentially localized perturbations.

Theorem 1.1. *Let $c_0 > 0$ and $a \in (0, \sqrt{c_0/2})$. Then there exist positive constants ε_0 and C satisfying the following: if $u(0, x) = \varphi_{c_0}(x - x_0) + v_0(x)$ and $\varepsilon := \|e^{ax}v_0\|_{L^2(\mathbb{R}^2)} + \|e^{ax}v_0\|_{L_y^1 L_x^2} + \|v_0\|_{L^2(\mathbb{R}^2)} < \varepsilon_0$, then there exist C^1 -functions $c(t, y)$ and $x(t, y)$ such that for $t \geq 0$,*

$$(1.5) \quad \|u(t, x, y) - \varphi_{c(t,y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon,$$

$$(1.6) \quad \sup_{y \in \mathbb{R}} (|c(t, y) - c_0| + |x_y(t, y)|) \leq C\varepsilon(1 + t)^{-1/2},$$

$$(1.7) \quad \|x_t(t, \cdot) - 2c(t, \cdot)\|_{L^2} \leq C\varepsilon(1 + t)^{-3/4},$$

$$(1.8) \quad \|e^{ax}(u(t, x + x(t, y), y) - \varphi_{c(t,y)}(x))\|_{L^2} \leq C\varepsilon(1 + t)^{-3/4}.$$

Remark 1.1. The KP-II equation has no localized solitary waves (see [6]). On the other hand, the KP-I equation has stable localized solitary waves (see [7, 22]) and line solitons of the KP-I equation are unstable ([33, 34, 42]).

The KP-II equation (1.1) is invariant under a change of variables

$$(1.9) \quad x \mapsto x + ky - 3k^2t + \gamma \quad \text{and} \quad y \mapsto y - 6kt \quad \text{for any } k, \gamma \in \mathbb{R},$$

and has a 3-parameter family of line soliton solutions

$$\mathcal{A} = \{\varphi_c(x + ky - (2c + 3k^2)t + \gamma) \mid c > 0, k, \gamma \in \mathbb{R}\}.$$

The set of all 1-soliton solutions of KdV or line soliton solutions of KP-II under the y -periodic boundary conditions are known to be stable in $L^2(\mathbb{R}^2)$ (see [24],[26]). However the set \mathcal{A} is not large enough to be stable for the flow generated by KP-II in $L^2(\mathbb{R}^2)$.

Theorem 1.2. *Let $c_0 > 0$. There exists a positive constant C such that for any $\varepsilon > 0$, there exists a solution of (1.1) such that $\|u(0, x, y) - \varphi_{c_0}(x)\|_{L^2} < \varepsilon$ and*

$$\liminf_{t \rightarrow \infty} t^{-1/4} \|u(t, x, y) - \varphi_{c_0}(x)\|_{L^2(\mathbb{R}^2)} \geq C\varepsilon.$$

Remark 1.2. If $(c, \gamma) \neq (c_0, 0)$, then $\|u(t, x, y) - \varphi_c(x - \gamma)\|_{L^2(\mathbb{R}^2)} = \infty$ thanks to the well-posedness result ([29]). Thus the *orbital instability*

$$\liminf_{t \rightarrow \infty} t^{-1/4} \inf_{v \in \mathcal{A}} \|u(t, \cdot) - v\|_{L^2(\mathbb{R}^2)} \geq C\varepsilon$$

follows immediately from Theorem 1.2.

Orbital instability is a consequence of finite speed propagations of local phase shifts along the crest of the modulating line soliton. We find that $c(t, y)$ and $\partial_y x(t, y)$ behave like a self-similar solution of the Burgers equation around $y = \pm \sqrt{8c_0 t}$.

Theorem 1.3. *Let $c_0 = 2$ and let v_0 and ε be the same as in Theorem 1.1. Then for any $R > 0$,*

$$\left\| \begin{pmatrix} c(t, \cdot) \\ x_y(t, \cdot) \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_B^+(t, y + 4t) \\ u_B^-(t, y - 4t) \end{pmatrix} \right\|_{L^2(|y \pm 4t| \leq R\sqrt{t})} = o(t^{-1/4})$$

as $t \rightarrow \infty$, where u_B^\pm are self similar solutions of the Burgers equation

$$\partial_t u = 2\partial_y^2 u \pm 4\partial_y(u^2)$$

such that

$$u_B^\pm(t, y) = \frac{\pm m_\pm H_{2t}(y)}{2(1 + m_\pm \int_0^y H_{2t}(y_1) dy_1)}, \quad H_t(y) = (4\pi t)^{-1/2} e^{-y^2/4t},$$

and that m_\pm are constants satisfying

$$\int_{\mathbb{R}} u_B^\pm(t, y) dy = \frac{1}{4} \int_{\mathbb{R}} c(0, y) dy + O(\varepsilon^2).$$

Now we recall known results on stability of planar traveling wave solutions. Stability of planar traveling waves in $L^2(\mathbb{R}^n)$ ($n \geq 2$) were studied for reaction diffusion equations by Xin ([41]), Levermore and Xin ([21]) and Kapitula ([18]). Stability of kink solutions of Hamiltonian systems has been studied for 3-dimensional ϕ^4 -model by Cuccagna ([9]).

The difficulty of those problems is that the spectrum of the linearized operator \mathcal{L} around planar traveling waves has continuous spectrum converging to 0 whereas in the case where $n = 1$, we see that 0 is an isolated eigenvalue of the linearized operator around the traveling wave solution and all the rest of the spectrum is in the left half plane and away from the imaginary axis. When $n \geq 2$, the paper [41] tells us that the semigroup generated by the linearized operator decays to zero like $t^{-(n-1)/4}$. This corresponds to the relation between our results and the asymptotic stability result for the KdV equation by Pego and Weinstein ([32]) where the spectrum of the linearized operator in $L^2(\mathbb{R}; e^{2ax} dx)$ consists of the isolated eigenvalue 0 and σ_c satisfying $\sigma_c \subset \{\lambda \in \mathbb{C} \mid \Re \lambda < -b\}$ for some $b > 0$. By measuring the size of perturbations with an exponentially weighted norm biased in the direction of motion, one obtains that exponential decay of the oscillating tail of the solution for both KdV and KP-II and that leads to exponential stability of the KdV 1-soliton. However, thanks to the tranverse direction, the linearized operator around a line soliton of the KP-II equation has two branches of continuous spectrum all the way up to 0 in $L^2(\mathbb{R}^2; e^{ax} dx dy)$ with $a > 0$. We remark that those resonant modes are exponentially growing as $x \rightarrow -\infty$ (see Lemma 2.1) and that

the corresponding continuous spectrum does not show up when we consider $L^2(\mathbb{R}^2)$ -linear stability of the line soliton. We refer the readers e.g. [1, 14] for linear stability of solitary waves and cnoidal waves to transverse perturbations.

Since the transverse direction is 1-dimensional, the rate of decay of $\|\partial_y^k c(t, \cdot)\|_{L^2}$ and $\|\partial_y^{k+1} x(t, \cdot)\|_{L^2}$ is at most $t^{-(2k+1)/4}$ and the nonlinearity of the modulation equations is quadratic, it was fortunate that they have the similar structure as the Burgers equations. Indeed, there are 1D-heat equations with quadratic nonlinearity whose solutions may not exist global in time ([10]).

Our plan of the present paper is as follows. In Section 2, we obtain explicit formula of resonant modes of \mathcal{L} and \mathcal{L}^* , where \mathcal{L} is a linearized operator of the KP-II equation around the line soliton $\varphi(x - 4t)$ by using the linearized Miura transformations. As is well known, the Miura transformations connect line solitons and the null solution of the KP-II equation with kink solutions of the modified KP-II equation and all the slowly decaying eigenmodes of the linearized equation $\partial_t u = \mathcal{L}u$ can be found by investigating the kernel and the cokernel of the linearized Miura transformation. We find two branches of (resonant) eigenmodes $\{g(x, \pm\eta)e^{iy\eta}\}$ of \mathcal{L} such that $g(x, \eta) \in L^2(\mathbb{R}; e^{2ax} dx)$ for a $a > 0$ and $\eta \in (-\eta_*, \eta_*)$, where η_* is a positive number depending on a . In Section 3, we prove that solutions which are orthogonal to resonant modes of \mathcal{L}^* decay exponentially in $L^2(\mathbb{R}^2; e^{2ax} dx dy)$ like solutions of the linearized KP-II equation around the null solution by using a bijection composed of the linearized Miura transformations by using an idea of [25]. In Section 4, we collect linear estimates of 1D damped wave equations which shall be used to analyze modulation equations of line solitons. In Section 5, we fix the decomposition (1.3) by imposing that $v(t)$ is orthogonal to secular resonant modes of \mathcal{L}^* . In Section 6, we derive modulation equations on $c(t, y)$ and $x(t, y)$ from the non-secular conditions introduced in Section 5. Since the KP equations are anisotropic in x and y , the resonant eigenfunctions cannot be written in the form $\{g(x)e^{iy\eta} \mid \eta \in \mathbb{R}\}$ as in the case for reaction diffusion equations ([18, 41]) or the ϕ^4 model ([9]). Moreover, the resonant eigenfunctions grow like $g(x, \eta) \sim e^{\eta^2|x|/2}$ as $x \rightarrow -\infty$. For this reason, we work on exponentially weighted space X and impose the non-secular conditions only for small η . To rewrite modulation equations of $c(t, y)$ and $x(t, y)$ in a PDE form, we compute the time derivative of the non-secular condition, take the inverse Fourier transform of the resulting equation. Although the modulation equations are non-local due to the η -dependence of the resonant modes $g(x, \eta)$, the dominant part of the modulation equations are damped wave equations. Indeed, the modulation equations for the line soliton $\varphi_{c_0}(x - 2c_0 t)$ with $c_0 = 2$ are approximately

$$(1.10) \quad \begin{pmatrix} b_t \\ \tilde{x}_t \end{pmatrix} \simeq \begin{pmatrix} 3\partial_y^2 & 8\partial_y^2 \\ 2 - \mu_3\partial_y^2 & \partial_y^2 \end{pmatrix} \begin{pmatrix} b \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} 6(bx_y)_y \\ 3(\tilde{x}_y)^2 - \frac{1}{4}b^2 \end{pmatrix},$$

where $\mu_3 = 1/2 + \pi^2/24$ and $b(t, y) = 4/3\{(c(t, y)/2)^{3/2} - 1\}$ (see (6.12) for the precise definition) and $\tilde{x}(t, y) = x(t, y) - 4t$. We remark that $\partial_t x(t, y) \simeq 2c(t, y)$ and $b(t, y) \simeq c(t, y) - 2 \rightarrow 0$ as $t \rightarrow \infty$. If we translate (1.10) into a system of $b(t, y)$ and $\partial_y x(t, y)$ and diagonalize the resulting equation, then we obtain a coupled Burgers equations.

In Section 7, we obtain $\mathcal{F}^{-1}L^\infty$ - L^2 decay estimates on $b(t, y)$ and $\partial_y x(t, y)$ presuming a decay estimate on $v(t)$ in $X := L^2(\mathbb{R}^2; e^{2ax} dx dy)$ and the L^2 -bound of $v(t)$. In Section 8, we prove the L^2 -estimate of v assuming the decay estimate on

$v(t)$ in X . In Section 9, we estimate the low-frequency part of v in $L^2(\mathbb{R}^2; e^{2ax} dx dy)$ by using the semigroup estimates obtained in Section 3. We estimate the high frequency part separately in Section 10 by using the virial type estimate to avoid the derivative loss. We remark that the potential term produced by the linearization around the line soliton is negligible to obtain time-global virial type estimates for the high frequency part. In Sections 11 and 12, we prove Theorems 1.1 and 1.2. In Section 13, we prove Theorem 1.3 by using a rescaling argument by Karch ([20]).

Using the inverse scattering method, Villarroel and Ablowitz ([39]) studied the Cauchy problem and stability of line solitons of the KP-II equation. However, it is not clear from their result that how modulations to line solitons evolve because they did not explain in which sense line solitons are stable. Moreover, our method does not rely on integrability of the equation except for the linear estimate and can possibly be applied to bidirectional models such as the Benney-Luke equation ([3, 31]). Our result is a first step toward $L^2(\mathbb{R}^2)$ -stability of planar solitary waves for non-integrable equations such as the generalized KP equations (see e.g. Martel-Merle [23] for H^1 -asymptotic stability of 1D solitary waves for gKdV).

Finally, let us introduce several notations. For Banach spaces V and W , let $B(V, W)$ be the space of all linear continuous operators from V to W and let $\|T\|_{B(V, W)} = \sup_{\|x\|_V=1} \|Tu\|_W$ for $T \in B(V, W)$. We abbreviate $B(V, V)$ as $B(V)$. For $f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathcal{S}'(\mathbb{R}^n)$, let

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx, \\ (\mathcal{F}^{-1}f)(x) = \check{f}(x) = \hat{f}(-x), \quad (m(D_x)f)(x) = (2\pi)^{-n/2} (\check{m} * f)(x).$$

We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \leq Cb$. Various constants will be simply denoted by C and C_i ($i \in \mathbb{N}$) in the course of the calculations. We denote $\langle x \rangle = \sqrt{1 + x^2}$ for $x \in \mathbb{R}$.

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2. THE MIURA TRANSFORMATION AND RESONANT MODES OF THE LINEARIZED OPERATOR

In this section, we will find resonant eigenmodes of the linearized operator around line solitons and prove exponential stability of non-resonant modes in an exponentially weighted space by using the linearized Miura transformations.

For $p \in [1, \infty]$ and $k \in \mathbb{N}$, let $L_a^p(\mathbb{R}) = \{v \mid e^{ax}v \in L^p(\mathbb{R})\}$ and $H_a^k(\mathbb{R}) = \{v \mid e^{ax}v \in H^k(\mathbb{R})\}$ whose norms are given by

$$\|v\|_{L_a^p(\mathbb{R})} = \|e^{ax}v\|_{L^p(\mathbb{R})}, \quad \|v\|_{H_a^k(\mathbb{R})} = \left(\sum_{j=0}^k \|\partial_x^j v\|_{L_a^2(\mathbb{R})}^2 \right)^{1/2}.$$

For any $a > 0$, we define the anti-derivative operator ∂_x^{-1} on $L_{\pm a}^2(\mathbb{R})$ by

$$\begin{aligned} (\partial_x^{-1}u)(x) &= -\int_x^\infty u(x_1) dx_1 \quad \text{for } u \in L_a^2(\mathbb{R}), \\ (\partial_x^{-1}u)(x) &= \int_{-\infty}^x u(x_1) dx_1 \quad \text{for } u \in L_{-a}^2(\mathbb{R}). \end{aligned}$$

The operator ∂_x^{-1} is bounded on $L_a^2(\mathbb{R})$. Indeed, it follows from Young's inequality that $\|\partial_x^{-1}\|_{B(L_a^2(\mathbb{R}))} = \|\partial_x^{-1}\|_{B(L_{-a}^2(\mathbb{R}))} = 1/a$ for $a > 0$.

We interpret (1.1) in the “integrated” form

$$(2.1) \quad \partial_t u + \partial_x^3 u + 3\partial_x^{-1}\partial_y^2 u + 3\partial_x(u^2) = 0,$$

where $\partial_x^{-1}\partial_y^2 u(x, y) = -\int_x^\infty \partial_y^2 u(x_1, y) dx_1$ in the sense of distribution, that is,

$$(2.2) \quad \langle \partial_x^{-1}\partial_y^2 u, \psi \rangle = -\int_{\mathbb{R}^2} \left(\int_x^\infty u(x_1, y) dx_1 \right) \partial_y^2 \psi(x, y) dx dy \quad \text{for } \psi \in C_0^\infty(\mathbb{R}^2).$$

If u is smooth and $u, \partial_y u \in X := L^2(\mathbb{R}^2; e^{2ax} dx dy)$, then

$$\langle \partial_x^{-1}\partial_y^2 u, \psi \rangle = \int_{\mathbb{R}^2} \left(\int_x^\infty \partial_y u(x_1, y) dx_1 \right) \partial_y \psi(x, y) dx dy.$$

Eq. (2.2) follows from the standard definition $\partial_x^{-1}\partial_y u = \mathcal{F}^{-1}(\frac{\eta}{\xi}\hat{u}(\xi, \eta))$ when a solution is exponentially localized in the x -direction. Indeed, we have $(\partial_x^{-1}u)(x, y) = (\mathcal{F}^{-1}(i\xi)^{-1}\hat{u})(x, y) = -\int_x^\infty u(x_1, y) dx_1$ for $u \in \{f \in C_0^\infty(\mathbb{R}^2) \mid \int_{\mathbb{R}} f(x, y) dx = 0 \ \forall y \in \mathbb{R}\} =: \mathcal{B}$. Since \mathcal{B} is a dense subset of $X \cap L^2(\mathbb{R}^2)$, we have (2.2) for $u \in X \cap L^2(\mathbb{R}^2)$ by taking a limit.

We remark that (2.1) has a solution in the class

$$u(t, x, y) - \varphi_c(x - 2ct) \in L_{loc}^\infty([0, \infty); X) \cap C(\mathbb{R}; L^2(\mathbb{R}^2))$$

for initial data $u(0) \in X \cap L^2(\mathbb{R}^2)$ (see Appendix E).

2.1. Resonant modes. Let $\varphi = \varphi_2$, $u(t, x, y) = \varphi(x - 4t) + U(t, x - 4t, y)$. Linearizing (2.1) around $U = 0$, we have

$$(2.3) \quad \partial_t U = \mathcal{L}U, \quad \mathcal{L}U = -\partial_x^3 U + 4\partial_x U - 3\partial_x^{-1}\partial_y^2 U - 6\partial_x(\varphi U).$$

Let $\mathcal{L}(\eta)u := -\partial_x^3 u + 4\partial_x u + 3\eta^2\partial_x^{-1}u - 6\partial_x(\varphi u)$ be the operator on $L_a^2(\mathbb{R})$ with its domain $D(\mathcal{L}(\eta)) = H_a^3(\mathbb{R})$. Since the potential of \mathcal{L} does not depend on y , we have $\mathcal{L}(u(x)e^{\pm i\eta y}) = e^{\pm i\eta y}\mathcal{L}(\eta)u(x)$. We will look for resonant modes $\{g(x, \eta)e^{i\eta y}\}$ such that $g(\cdot, \eta) \in L_a^2(\mathbb{R})$ is a solution of $\mathcal{L}(\eta)u = \lambda u$.

Lemma 2.1. *Let $\eta \in \mathbb{R} \setminus \{0\}$, $\beta(\eta) = \sqrt{1 + i\eta}$, $\lambda(\eta) = 4i\eta\beta(\eta)$ and*

$$g(x, \eta) = \frac{-i}{2\eta\beta(\eta)}\partial_x^2(e^{-\beta(\eta)x} \operatorname{sech} x), \quad g^*(x, \eta) = \partial_x(e^{\beta(-\eta)x} \operatorname{sech} x).$$

Then

$$(2.4) \quad \mathcal{L}(\eta)g(x, \pm\eta) = \lambda(\pm\eta)g(x, \pm\eta),$$

$$(2.5) \quad \mathcal{L}(\eta)^*g^*(x, \pm\eta) = \lambda(\mp\eta)g^*(x, \pm\eta),$$

$$(2.6) \quad \int_{\mathbb{R}} g(x, \eta)\overline{g^*(x, \eta)} dx = 1, \quad \int_{\mathbb{R}} g(x, \eta)\overline{g^*(x, -\eta)} dx = 0.$$

Lemma 2.1 will be proved in Subsection 2.2.

To resolve the singularity of $g(x, \eta)$ and the degeneracy of $g_*(x, \eta)$ at $\eta = 0$, we decompose resonant modes and adjoint resonant modes into their real parts and imaginary parts. Let

$$\begin{aligned} g_1(x, \eta) &= g(x, \eta) + g(x, -\eta), & g_2(x, \eta) &= i\eta\{g(x, \eta) - g(x, -\eta)\}, \\ g_1^*(x, \eta) &= \frac{1}{2}\{g^*(x, \eta) + g^*(x, -\eta)\}, & g_2^*(x, \eta) &= \frac{i}{2\eta}\{g^*(x, \eta) - g^*(x, -\eta)\}. \end{aligned}$$

Then we have the following.

Lemma 2.2.

$$\int_{\mathbb{R}} g_j(x, \eta) \overline{g_k^*(x, \eta)} dx = \delta_{jk} \quad \text{for } j, k = 1, 2.$$

$$\begin{aligned} \mathcal{L}(\eta)g_1(x, \eta) &= \Re\lambda(\eta)g_1(x, \eta) + \frac{\Im\lambda(\eta)}{\eta}g_2(x, \eta), \\ \mathcal{L}(\eta)g_2(x, \eta) &= -\eta\Im\lambda(\eta)g_1(x, \eta) + \Re\lambda(\eta)g_2(x, \eta), \\ \mathcal{L}(\eta)^*g_1^*(x, \eta) &= \Re\lambda(\eta)g_1^*(x, \eta) - \eta\Im\lambda(\eta)g_2^*(x, \eta), \\ \mathcal{L}(\eta)^*g_2^*(x, \eta) &= \frac{\Im\lambda(\eta)}{\eta}g_1^*(x, \eta) + \Re\lambda(\eta)g_2^*(x, \eta). \end{aligned}$$

Proof. Lemma 2.2 follows immediately from Lemma 2.1 since $\overline{\lambda(\eta)} = \lambda(-\eta)$, $\overline{g(x, \eta)} = g(x, -\eta)$ and $\overline{g^*(x, \eta)} = g^*(x, -\eta)$. \square

We remark that $\mathcal{L}(0)$ coincides with the linearized operator of the KdV equation around the 1-soliton $\varphi(x-4t)$ and that $g_k(x, 0) \in \ker_g(\mathcal{L}(0))$, $g_k^*(x, 0) \in \ker_g(\mathcal{L}(0)^*)$ for $k = 1$ and 2 .

Claim 2.1. *Let $a \in (0, 2)$ and $\nu(\eta) = \Re\beta(\eta) - 1$. Let η_0 be a positive number such that $\nu_0 := \nu(\eta_0) < a$. Then for $\eta \in [-\eta_0, \eta_0]$,*

$$\begin{aligned} g_1(x, \eta) &= \frac{1}{4}\varphi' + \frac{x}{4}\varphi' + \frac{1}{2}\varphi + O(\eta^2), & g_2(x, \eta) &= -\frac{1}{2}\varphi' + O(\eta^2) \quad \text{in } L_a^2(\mathbb{R}), \\ g_1^*(x, \eta) &= \frac{1}{2}\varphi + O(\eta^2), & g_2^*(x, \eta) &= \int_{-\infty}^x \partial_c \varphi dx + O(\eta^2) \quad \text{in } L_{-a}^2(\mathbb{R}), \end{aligned}$$

where $\partial_c \varphi = \partial_c \varphi_c|_{c=2}$.

Proof. Since $g_1(x, \eta)$ and $g_2(x, \eta)$ are even in η ,

$$\begin{aligned} g_1(x, \eta) &= \frac{1}{2i\eta} \partial_x^2 \left\{ \left(\frac{e^{-\beta(\eta)x}}{\beta(\eta)} - \frac{e^{-\beta(-\eta)x}}{\beta(-\eta)} \right) \operatorname{sech} x \right\} \\ &= \partial_s \left(\frac{e^{-\sqrt{s}x}}{\sqrt{s}} \operatorname{sech} x \right) \Big|_{s=1} + O(\eta^2) \\ &= \frac{x+1}{4}\varphi' + \frac{1}{2}\varphi + O(\eta^2), \end{aligned}$$

and

$$g_2(x, \eta) = (e^{-x} \operatorname{sech} x)_{xx} + O(\eta^2) = -\frac{1}{2}\varphi' + O(\eta^2).$$

We can compute $g_1^*(x, \eta)$ and $g_2^*(x, \eta)$ in the same way. \square

2.2. Linearized Miura transformation. Now we recall the Miura transformation of the KP-II equation. Let

$$M_{\pm}^c(v) = \pm \partial_x v + \partial_x^{-1} \partial_y v - v^2 + \frac{c}{2}.$$

The transformations M_{\pm}^c relate the KP-II equation to the mKP-II equation (mKP-II) which reads

$$(2.7) \quad \partial_t v + \partial_x^3 v + 3\partial_x^{-1} \partial_y^2 v - 6v^2 \partial_x v + 6\partial_x v \partial_x^{-1} \partial_y v = 0.$$

Formally, if $v(t, x, y)$ is a solution of (2.7) and $c > 0$, then $M_{\pm}^c(v)(t, x - 3ct, y)$ are solutions of the KP-II equation (1.1). A line soliton solution $\varphi_c(x - 2ct)$ of the KP-II equation is related to a kink solution $Q_c(x + ct)$ of (2.7), where $Q_c(x) = \sqrt{\frac{c}{2}} \tanh\left(\sqrt{\frac{c}{2}} x\right)$. Indeed, we have

$$(2.8) \quad M_+^c(Q_c) = \varphi_c, \quad M_-^c(Q_c) = 0.$$

From now on, let $c = 2$, $Q = Q_2$ and $M_{\pm} = M_{\pm}^2$. Let $v(t, x, y) = Q(x + 2t) + V(t, x + 2t, y)$ and linearize (2.7) around $V = 0$. Then

$$(2.9) \quad \begin{aligned} \partial_t V &= \mathcal{L}_M V, \\ \mathcal{L}_M V &:= -\partial_x^3 V - 2\partial_x V - 3\partial_x^{-1} \partial_y^2 V + 6\partial_x(Q^2 V) - 6Q' \partial_x^{-1} \partial_y V \\ &= -\partial_x^3 V + 4\partial_x V - 3\partial_x^{-1} \partial_y^2 V - 6\partial_x(Q' V) - 6Q' \partial_x^{-1} \partial_y V. \end{aligned}$$

In the last line, we use $Q' = 1 - Q^2$. Let X_M be the Banach space equipped with the norm $\|v\|_{X_M} := (\|v\|_X^2 + \|\partial_x v\|_X^2 + \|\partial_x^{-1} \partial_y v\|_X^2)^{1/2}$. Thanks to the smoothing effect of \mathcal{L}_0 in X (see Lemma 3.4 in Section 3), the initial value problem

$$\partial_t v = \mathcal{L}_M v, \quad v(0) = v_0$$

has a unique solution in the class $C([0, \infty); X_M)$.

Solutions of (2.3) are related to those of (2.9) by the linearized Miura transformation

$$(2.10) \quad u = \nabla M_+(Q)v = \partial_x v + \partial_x^{-1} \partial_y v - 2Qv.$$

Another linearized Miura transformation

$$(2.11) \quad u = \nabla M_-(Q)v = -\partial_x v + \partial_x^{-1} \partial_y v - 2Qv$$

maps solutions of (2.9) to those of the linearized KP-II around 0

$$(2.12) \quad \partial_t u + \partial_x^3 u - 4\partial_x u + 3\partial_x^{-1} \partial_y^2 u = 0.$$

Lemma 2.3. *Suppose that v is a solution to (2.9). Then $u = \nabla M_+(Q)v$ satisfies (2.3) and $u = \nabla M_-(Q)v$ satisfies (2.12).*

Proof of Lemma 2.3. By a straightforward computation, we find that

$$(2.13) \quad \mathcal{L} \nabla M_+(Q) = \nabla M_+(Q) \mathcal{L}_M, \quad \mathcal{L}_0 \nabla M_-(Q) = \nabla M_-(Q) \mathcal{L}_M.$$

Let $u_{\pm} = \nabla M_{\pm}(Q)v$. Then it follows from (2.13) that

$$\begin{aligned} \partial_t u_+ - \mathcal{L} u_+ &= \nabla M_+(Q)(\partial_t v - \mathcal{L}_M v), \\ \partial_t u_- - \mathcal{L}_0 u_- &= \nabla M_-(Q)(\partial_t v - \mathcal{L}_M v). \end{aligned}$$

Therefore u_+ and u_- are solutions of (2.3) and (2.12), respectively, if v is a solution to (2.9). Thus we complete the proof. \square

Lemma 2.4. *Let $a > 0$ and $v(t) \in C([0, \infty); X_M)$ be a solution to (2.9).*

- (1) Suppose that $u(t) \in C([0, \infty); X)$ is a solution to (2.3) satisfying $u(0) = \nabla M_+(Q)v(0)$. Then $u(t) = \nabla M_+(Q)v(t)$ holds for every $t \geq 0$.
(2) Suppose that $u(t) \in C([0, \infty); X)$ is a solution to (2.12) satisfying $u(0) = \nabla M_-(Q)v(0)$. Then $u(t) = \nabla M_-(Q)v(t)$ holds for every $t \geq 0$.

Proof. Let $\tilde{u}(t) = \nabla M_+(Q)v(t)$. Then $\tilde{u}(t) \in C([0, \infty); X)$. Moreover, Lemma 2.3 implies that $\tilde{u}(t)$ is a solution of (2.3). Since $u(0) = \tilde{u}(0)$ and both $u(t)$ and $\tilde{u}(t)$ are solutions of (2.3) in the class $C([0, \infty); X)$, we have $u(t) = \tilde{u}(t)$. Thus we prove (i). We can prove (ii) in exactly the same way. This completes the proof of Lemma 2.4. \square

Let

$$\mathcal{L}_M(\eta)v := -\partial_x^3 v + 4\partial_x v + 3\eta^2 \partial_x^{-1} v - 6\partial_x(Q'v) - 6i\eta Q' \partial_x^{-1} v.$$

Then $\mathcal{L}_M(v(x)e^{i\eta y}) = e^{i\eta y} \mathcal{L}_M(\eta)v(x)$ and $\mathcal{L}_M(\eta)$ has the following resonant modes.

Lemma 2.5. *Let $\eta \in \mathbb{R}$ and*

$$g_M(x, \eta) = \frac{-1}{2\beta(\eta)} \partial_x (e^{-\beta(\eta)x} \operatorname{sech} x), \quad g_M^*(x, \eta) = e^{\beta(-\eta)x} \operatorname{sech} x.$$

Then

$$(2.14) \quad \mathcal{L}_M(\eta)g_M(x, \eta) = \lambda(\eta)g_M(x, \eta),$$

$$(2.15) \quad \mathcal{L}_M(\eta)^* g_M^*(x, \eta) = \lambda(-\eta)g_M^*(x, \eta),$$

$$(2.16) \quad \int_{\mathbb{R}} g_M(x, \eta) \overline{g_M^*(x, \eta)} dx = 1.$$

The eigenvalue problem $\mathcal{L}u = \lambda u$ is related to the eigenvalue problem $\mathcal{L}_M v = \lambda v$ via (2.13). Before we prove Lemmas 2.1 and 2.5, we will investigate the kernel and the cokernel of bounded operators $\mathcal{M}_{\pm}(\eta) : H_a^1(\mathbb{R}) \rightarrow L_a^2(\mathbb{R})$ defined by

$$\mathcal{M}_{\pm}(\eta)g(x) := \pm g'(x) - i\eta \int_x^{\infty} g(t) dt - 2Q(x)g(x).$$

Lemma 2.6. *Let $a \in (0, 2)$ and η_0 be a positive number satisfying $a > \nu_0$. Then $\ker(\mathcal{M}_-(\eta)) = \operatorname{span}\{g_M(\cdot, \eta)\}$ and $\operatorname{Range}(\mathcal{M}_-(\eta)) = L_a^2(\mathbb{R})$. Moreover, for any $\eta \in [-\eta_0, \eta_0]$ and $f \in L_a^2(\mathbb{R})$, there exists a unique solution $v \in H_a^1(\mathbb{R})$ of*

$$(2.17) \quad \mathcal{M}_-(\eta)v = f,$$

that satisfies $\int_{\mathbb{R}} v(x) \overline{g_M^(x, \eta)} dx = 0$. Moreover,*

$$\|v\|_{H_a^1(\mathbb{R})} + |\eta| \|\partial_x^{-1} v\|_{L_a^2(\mathbb{R})} \leq \frac{C}{a - \nu(\eta_0)} \|f\|_{L_a^2(\mathbb{R})},$$

where C is a constant depending only on a .

Lemma 2.7. *Let $a \in (0, 2)$ and η_0 be a positive number satisfying $a > \nu_0$. If $\eta \in [-\eta_0, \eta_0]$, then $\ker(\mathcal{M}_+(\eta)) = \{0\}$ and $\operatorname{Range}(\mathcal{M}_+(\eta)) = {}^{\perp} \operatorname{span}\{g^*(x, -\eta)\}$. Moreover, for any $f \in L_a^2(\mathbb{R})$ satisfying $\int_{\mathbb{R}} f(x) \overline{g^*(x, -\eta)} dx = 0$, there exists a unique solution $v \in H_a^1(\mathbb{R})$ of*

$$(2.18) \quad \mathcal{M}_+(\eta)v = f,$$

satisfying

$$(2.19) \quad \|v\|_{H_a^1(\mathbb{R})} + |\eta| \|\partial_x^{-1} v\|_{L_a^2(\mathbb{R})} \leq C \|f\|_{L_a^2(\mathbb{R})},$$

where C is a constant depending only on a . If f satisfies $\int_{\mathbb{R}} f(x) \overline{g^*(x, \eta)} dx = 0$ in addition, then $\int_{\mathbb{R}} v(x) \overline{g_M^*(x, \eta)} dx = 0$.

Proof of Lemma 2.6. Suppose $v \in \ker(\mathcal{M}_-(\eta))$. Then $v \in H_a^1(\mathbb{R})$ and

$$(2.20) \quad -v'' + i\eta v - 2(Qv)' = 0.$$

Eq. (2.20) has a fundamental system $\{\tilde{g}_1, \tilde{g}_2\}$, where

$$\tilde{g}_1(x) = \left(e^{-\beta(\eta)x} \operatorname{sech} x \right)_x, \quad \tilde{g}_2(x) = \left(e^{\beta(\eta)x} \operatorname{sech} x \right)_x.$$

Since $1 \leq \Re \beta(\nu) \leq \Re \beta(\eta_0)$ and

$$(2.21) \quad \tilde{g}_1(x) \sim e^{-(\beta(\eta) \pm 1)x} \quad \text{and} \quad \tilde{g}_2(x) \sim e^{(\beta(\eta) \mp 1)x} \quad \text{as } x \rightarrow \pm\infty,$$

it follows that $\tilde{g}_1 \in H_a^1(\mathbb{R})$ and $\tilde{g}_2 \notin H_a^1(\mathbb{R})$ and that $v(x) = \alpha \tilde{g}_1(x)$ for an α . Thus we prove $\ker(\mathcal{M}_-(\eta)) = \operatorname{span}\{g_M(\cdot, \eta)\}$.

Suppose $v \in H_a^1(\mathbb{R})$ is a solution of (2.17). Then v satisfies an ODE

$$(2.22) \quad -v'' + i\eta v - 2(Qv)' = f'.$$

By the variation of the constants formula,

$$\begin{aligned} v(x) &= \tilde{g}_1(x) \int^x \frac{\tilde{g}_2(t) f'(t)}{W(t)} dt - \tilde{g}_2(x) \int^x \frac{\tilde{g}_1(t) f'(t)}{W(t)} dt \\ &= \tilde{g}_1(x) \int^x k_1'(t) f(t) dt + \tilde{g}_2(x) \int^x k_2'(t) f(t) dt, \end{aligned}$$

where $W(t) = \tilde{g}_1(t) \tilde{g}_2'(t) - \tilde{g}_1'(t) \tilde{g}_2(t) = -2i\eta\beta(\eta) \operatorname{sech}^2 t$,

$$\begin{aligned} k_1(t) &= -\frac{\tilde{g}_2(t)}{W(t)} = \frac{e^{\beta(\eta)t} (\beta(\eta) \cosh t - \sinh t)}{2i\eta\beta(\eta)}, \\ k_2(t) &= \frac{\tilde{g}_1(t)}{W(t)} = \frac{e^{-\beta(\eta)t} (\beta(\eta) \cosh t + \sinh t)}{2i\eta\beta(\eta)}, \end{aligned}$$

$k_1'(t) = (2\beta(\eta))^{-1} e^{\beta(\eta)t} \cosh t$ and $k_2'(t) = -(2\beta(\eta))^{-1} e^{-\beta(\eta)t} \cosh t$. Now let

$$(2.23) \quad \begin{aligned} v(x) &= \alpha \tilde{g}_1(x) + T_1(f) + T_2(f), \\ T_1(f) &= -\tilde{g}_1(x) \int_x^\infty k_1'(t) f(t) dt, \quad T_2(f) = -\tilde{g}_2(x) \int_x^\infty k_2'(t) f(t) dt, \end{aligned}$$

where α is a constant to be chosen later. Since $\operatorname{sech} x \cosh t \leq e^{t-x}$ for $t \in [x, \infty)$ and $\nu(\eta) \leq \nu_0$ for $\eta \in [-\eta_0, \eta_0]$,

$$|\tilde{g}_1(x) k_1'(t)| \lesssim e^{\nu_0(t-x)} \quad \text{if } t \geq x.$$

Using Young's inequality and the above, we have

$$\begin{aligned} \|T_1(f)\|_{L_a^2(\mathbb{R})} &\lesssim \left\| \int_x^\infty e^{\nu(\eta)(t-x)} |f(t)| dt \right\|_{L_a^2(\mathbb{R})} \\ &\lesssim \|e^{-(a-\nu_0)t}\|_{L^1(0,\infty)} \|f\|_{L_a^2(\mathbb{R})} \leq \frac{C_0}{a-\nu_0} \|f\|_{L_a^2(\mathbb{R})}, \end{aligned}$$

where C_0 is a constant independent of η_0 and $f \in L_a^2(\mathbb{R})$. Using the fact that $0 \leq \cosh t \operatorname{sech} x \leq e^{t-x}$ if $x \leq t$ and that $\nu(\eta) \geq 0$, we have

$$\begin{aligned} \|T_2(f)\|_{L_a^2(\mathbb{R})} &\lesssim \left\| \int_x^\infty e^{\nu(\eta)(x-t)} |f(t)| dt \right\|_{L_a^2(\mathbb{R})} \\ &\lesssim \|e^{-(a+\nu(\eta))t}\|_{L^1(0,\infty)} \|f\|_{L_a^2(\mathbb{R})} \leq C_1 \|f\|_{L_a^2(\mathbb{R})}, \end{aligned}$$

where C_1 is a constant independent of η_0 and $f \in L_a^2(\mathbb{R})$. Since

$$\int_{\mathbb{R}} \tilde{g}_1(x) \overline{g_M^*(x, \eta)} dx = - \int_{\mathbb{R}} \operatorname{sech}^2 x (\beta(\eta) - \tanh x) dx = -2\beta(\eta) \neq 0,$$

there exists a unique α such that $\int v(x) \overline{g^*(x, \eta)} dx = 0$. Since $L_a^2(\mathbb{R}) \ni f \mapsto T_1(f)$, $T_2(f) \in L_a^2(\mathbb{R})$ are continuous, $\alpha = \alpha(f)$ is also continuous in f . Thus we prove that there exists a constant C_2 such that

$$(2.24) \quad \|v\|_{L_a^2(\mathbb{R})} \leq C_2 \|f\|_{L_a^2(\mathbb{R})}$$

for every $\eta \in [-\eta_0, \eta_0] \setminus \{0\}$ and $f \in L_a^2(\mathbb{R})$.

Differentiating (2.23) with respect to x , we have

$$v'(x) = \alpha \tilde{g}_1'(x) - f(x) - \tilde{g}_1'(x) \int_x^\infty k_1'(t) f(t) dt - \tilde{g}_2'(x) \int_x^\infty k_2'(t) f(t) dt.$$

We can prove

$$(2.25) \quad \|v'(x)\|_{L_a^2(\mathbb{R})} \leq \frac{C_3}{a - \nu_0} \|f\|_{L_a^2(\mathbb{R})},$$

in the same way as (2.24), where C_3 is a positive constant independent of η_0 and $f \in L_a^2(\mathbb{R})$. Combining (2.24) and (2.25) with (2.17), we have

$$\|\eta\| \|\partial_x^{-1} v\|_{L_a^2(\mathbb{R})} \leq \|v'\|_{L_a^2(\mathbb{R})} + 2\|Qv\|_{L_a^2(\mathbb{R})} + \|f\|_{L_a^2(\mathbb{R})} \leq \frac{C_4}{a - \nu_0} \|f\|_{L_a^2(\mathbb{R})},$$

where C_4 is a positive constant independent of η_0 and $f \in L_a^2(\mathbb{R})$. Thus we complete the proof. \square

Proof of Lemma 2.7. First, we will show that $\ker(\mathcal{M}_+(\eta)^*) = \operatorname{span}\{\tilde{g}_2(x)\}$. Since $\mathcal{M}_-(\eta)$ is formally an adjoint of $\mathcal{M}_+(\eta)$, we easily see that $h \in \ker(\mathcal{M}_+(\eta)^*) \subset L_a^2(\mathbb{R})$ is a solution of (2.22) and that $h(x) = \alpha \tilde{g}_2(x) = \alpha g^*(x, -\eta)$ for an $\alpha \in \mathbb{C}$. Since $\ker(\mathcal{M}_+(\eta)^*) = \operatorname{span}\{\tilde{g}_2(x)\}$, we have $\operatorname{Range}(\mathcal{M}_+(\eta)) \subset {}^\perp \operatorname{span}\{g^*(x, -\eta)\}$.

Next we will show that $\ker(\mathcal{M}_+(\eta)) = \{0\}$. Suppose $\mathcal{M}_+(\eta)h = 0$. Then

$$(2.26) \quad h'' - 2(Qh)' + i\eta h = 0.$$

Eq. (2.26) has a fundamental system $\{h_1(x), h_2(x)\}$, where

$$h_1(x) = e^{\beta(-\eta)x} \cosh x, \quad h_2(x) = e^{-\beta(-\eta)x} \cosh x.$$

Since

$$(2.27) \quad h_1(x) \sim e^{(\beta(-\eta) \pm 1)x}, \quad h_2(x) \sim e^{(-\beta(-\eta) \pm 1)x} \quad \text{as } x \rightarrow \pm\infty,$$

it is clear that $h \in H_a^1(\mathbb{R})$ if and only if $h = 0$. Thus we prove $\ker(\mathcal{M}_+(\eta)) = \{0\}$.

Secondly, we will show that $\operatorname{Range}(\mathcal{M}_+(\eta)) = {}^\perp \operatorname{span}\{g^*(x, -\eta)\}$. Suppose that $v \in H_a^1(\mathbb{R})$ is a solution of (2.18). Then

$$(2.28) \quad v'' - 2(Qvg)' + i\eta v = f'.$$

By the variation of constants formula, we can find the following solution of (2.28).

$$(2.29) \quad v(x) = T_3(f) + T_4(f),$$

$$(2.30) \quad \begin{aligned} T_3(f) &:= \frac{e^{\beta(-\eta)x} \cosh x}{2\beta(-\eta)} \int_x^\infty \left(e^{-\beta(-\eta)t} \operatorname{sech} t \right)_t f(t) dt, \\ T_4(f) &:= \frac{e^{-\beta(-\eta)x} \cosh x}{2\beta(-\eta)} \int_{-\infty}^x \left(e^{\beta(-\eta)t} \operatorname{sech} t \right)_t f(t) dt. \end{aligned}$$

Since $\cosh x \operatorname{sech} t \leq e^{|x-t|}$, we have

$$\begin{aligned} \|T_3(f)\|_{L_a^2(\mathbb{R})} &\lesssim \left\| \int_x^\infty e^{\nu(-\eta)(x-t)} |f(t)| dt \right\|_{L_a^2(\mathbb{R})} \\ &\lesssim \|e^{-(a+\nu(-\eta))t}\|_{L^1(0,\infty)} \|f\|_{L_a^2(\mathbb{R})} \leq C_1 \|f\|_{L_a^2(\mathbb{R})}, \end{aligned}$$

where C_1 is a constant depending only on a . If $\int_{\mathbb{R}} f(x) \overline{g^*(x, -\eta)} dx = 0$, then $T_4(f)$ can be rewritten as

$$(2.31) \quad T_4(f) = -\frac{e^{-\beta(-\eta)x} \cosh x}{2\beta(-\eta)} \int_x^\infty \left(e^{\beta(-\eta)t} \operatorname{sech} t \right)_t f(t) dt.$$

Using (2.30) for $x \geq 0$ and (2.31) for $x \leq 0$ and the fact that $\cosh x \operatorname{sech} t \leq 2e^{-|x-t|}$ for t satisfying $|t| \geq |x|$, we have

$$\begin{aligned} \|T_4(f)\|_{L_a^2(\mathbb{R})} &\lesssim \left\| \int_x^\infty e^{(a-\nu(-\eta))(x-t)} e^{at} |f(t)| dt \right\|_{L^1(0,\infty)} \\ &\quad + \left\| \int_{-\infty}^x e^{(a-\nu(-\eta)-2)(x-t)} e^{at} |f(t)| dt \right\|_{L^1(-\infty,0)} \\ &\leq C_2 \|f\|_{L_a^2(\mathbb{R})}, \end{aligned}$$

where C_2 is a constant depending only on a . Thus we prove that (2.18) has a unique solution $v \in L_a^2(\mathbb{R})$. We can prove (2.19) in the same way as Lemma 2.6.

Suppose f satisfies $\int_{\mathbb{R}} f(x) \overline{g^*(x, \pm\eta)} dx = 0$, then it follows from (2.35) and (2.18) that

$$\begin{aligned} 2i\eta \int_{\mathbb{R}} v(x) \overline{g_M^*(x, \eta)} dx &= - \int_{\mathbb{R}} \mathcal{M}_+(\eta) v(x) \overline{g^*(x, \eta)} dx \\ &= - \int_{\mathbb{R}} f(x) \overline{g^*(x, \eta)} dx = 0. \end{aligned}$$

Thus we have $\int_{\mathbb{R}} v(x) \overline{g_M^*(x, \eta)} dx = 0$ for $\eta \in [-\eta_0, \eta_0] \setminus \{0\}$. This completes the proof of Lemma 2.7. \square

Now we are in position to prove Lemmas 2.1 and 2.5.

Proof of Lemmas 2.1 and 2.5. First, we will show that $\nabla M_+(Q) g_M(x, \eta) e^{i\eta y}$ are the resonant eigenmodes of \mathcal{L} and that $\nabla M_-(Q) (g_M(x, \eta) e^{i\eta y}) = 0$ by using (2.13),

Let $\mathcal{L}_0(\eta)u := -\partial_x^3 u + 4\partial_x u + 3\eta^2 \partial_x^{-1} u$ be the operator on $L_a^2(\mathbb{R})$ with its domain $D(\mathcal{L}_0(\eta)) = H_a^3(\mathbb{R})$. By the definition of $\mathcal{L}(\eta)$ and $\mathcal{M}_\pm(\eta)$, we have $\mathcal{L}_0(u(x) e^{\pm i\eta y}) = e^{\pm i\eta y} \mathcal{L}_0(\eta)u(x)$ and

$$\nabla M_\pm(Q)(g(x) e^{i\eta y}) = (\mathcal{M}_\pm(\eta)g)(x) e^{i\eta y}.$$

In view of (2.13),

$$(2.32) \quad \mathcal{L}(\eta)\mathcal{M}_+(\eta) = \mathcal{M}_+(\eta)\mathcal{L}_M(\eta),$$

$$(2.33) \quad \mathcal{L}_0(\eta)\mathcal{M}_-(\eta) = \mathcal{M}_-(\eta)\mathcal{L}_M(\eta).$$

By a simple computation, we find

$$(2.34) \quad \mathcal{M}_+(\eta)g_M(x, \eta) = -2i\eta g(x, \eta), \quad \mathcal{M}_-(\eta)g_M(x, \eta) = 0.$$

Combining (2.33) and (2.34), we have

$$\mathcal{M}_-(\eta)\mathcal{L}_M(\eta)g_M(x, \eta) = \mathcal{L}_0(\eta)\mathcal{M}_-(\eta)g_M(x, \eta) = 0.$$

Since $g_M(x, \eta) \in H_a^4(\mathbb{R})$ for an $a \in (\nu(\eta), 2)$, we have $\mathcal{L}_M(\eta)g_M(x, \eta) \in H_a^1(\mathbb{R})$ and $\mathcal{L}_M(\eta)g_M(x, \eta) \in \ker \mathcal{M}_-(\eta)$. Lemma 2.6 implies that there exists a $\lambda(\eta) \in \mathbb{C}$ such that $\mathcal{L}_M(\eta)g_M(x, \eta) = \lambda(\eta)g_M(x, \eta)$. Since $g_M(x, \eta) \sim e^{-(1+\beta(\eta))x}$ as $x \rightarrow \infty$, we see that

$$\lambda(\eta) = (1 + \beta(\eta))^3 - 4(1 + \beta(\eta)) - \frac{3\eta^2}{1 + \beta(\eta)} = 4i\eta\beta(\eta).$$

Thus we prove (2.14). It follows from (2.14), (2.32) and (2.34) that

$$\begin{aligned} \mathcal{L}(\eta)g(x, \eta) &= \frac{i}{2\eta}\mathcal{M}_+(\eta)\mathcal{L}_M(\eta)g_M(x, \eta) \\ &= \frac{i\lambda(\eta)}{2\eta}\mathcal{M}_+(\eta)g_M(x, \eta) = \lambda(\eta)g(x, \eta), \end{aligned}$$

and $\mathcal{L}(\eta)g(x, -\eta) = \overline{\mathcal{L}(\eta)g(x, \eta)} = \lambda(-\eta)g(x, -\eta)$.

Using the fact that $\partial_x \mathcal{L}(\eta)^* = -\mathcal{L}(-\eta)\partial_x$ (formally) and φ is even, we can easily confirm (2.5). Since $g_M(x, \eta)$ is a solution of (2.20) and

$$g^*(x, \eta) = -2\beta(-\eta)g_M(-x, -\eta), \quad Q(-x) = -Q(x),$$

we have $\partial_x^2 g^*(x, \eta) + 2\partial_x(Q(x)g^*(x, \eta)) + i\eta g^*(x, \eta) = 0$. Combining the above with $g^*(x, \eta) = \partial_x g_M^*(x, \eta)$, we have

$$(2.35) \quad \begin{aligned} \mathcal{M}_+(\eta)^* g^*(x, \eta) &= -\partial_x g^*(x, \eta) + i\eta \int_{-\infty}^x g^*(t, \eta) dt - 2Q(x)g^*(x, \eta) \\ &= 2i\eta g_M^*(x, \eta). \end{aligned}$$

By (2.32),

$$(2.36) \quad \mathcal{M}_+(\eta)^* \mathcal{L}(\eta)^* = \mathcal{L}_M(\eta)^* \mathcal{M}_+(\eta)^*.$$

Eq. (2.15) follows immediately from (2.5), (2.35) and (2.36).

Next, we will prove (2.16). By integration by parts,

$$\int g_M(x, \eta) \overline{g_M^*(x, \eta)} dx = \frac{1}{2\beta(\eta)} \int (\beta(\eta) + \tanh x) \operatorname{sech}^2 x dx = 1.$$

Finally, we will prove (2.6). By (2.34) and (2.35),

$$\begin{aligned} 2i\eta \int_{\mathbb{R}} g(x, \eta) \overline{g^*(x, \eta)} dx &= - \int_{\mathbb{R}} \mathcal{M}_+(\eta) g_M(x, \eta) \overline{g^*(x, \eta)} dx \\ &= - \int_{\mathbb{R}} g_M(x, \eta) \overline{\mathcal{M}_+^* g^*(x, \eta)} dx \\ &= 2i\eta \int_{\mathbb{R}} g_M(x, \eta) \overline{g_M^*(x, \eta)} dx = 2i\eta. \end{aligned}$$

Thus we prove (2.6) for $\eta \neq 0$. \square

If η is large, the operators $\mathcal{M}_{\pm}(\eta) : H_a^1(\mathbb{R}) \rightarrow L_a^2(\mathbb{R})$ have bounded inverse.

Lemma 2.8. *Suppose $a \in (0, 2)$, $\eta > 0$ and $\nu(\eta) > a$.*

(1) *For every $f \in L_a^2(\mathbb{R})$, there exists a unique solution v_+ of (2.17) satisfying*

$$(2.37) \quad \|v_+\|_{H_a^1(\mathbb{R})} + |\eta| \|\partial_x^{-1} v_+\|_{L_a^2(\mathbb{R})} \leq \frac{C}{\nu(\eta) - a} \|f\|_{L_a^2(\mathbb{R})},$$

where C is a constant depending only on a .

(2) *For every $f \in L_a^2(\mathbb{R})$, there exists a unique solution v_- of (2.18) satisfying*

$$(2.38) \quad \|v_-\|_{H_a^1(\mathbb{R})} + |\eta| \|\partial_x^{-1} v_-\|_{L_a^2(\mathbb{R})} \leq \frac{C}{\nu(\eta) - a} \|f\|_{L_a^2(\mathbb{R})},$$

where C is a constant depending only on a .

Proof of Lemma 2.8. If $\nu(\eta) > a > 0$, then (2.21) and (2.27) imply $\ker(\mathcal{M}_{\pm}(\eta)) = \{0\}$ and that (2.17) and (2.18) have at most one solution.

First we prove (1). Let

$$v_+(x) = \tilde{g}_1(x) \int_{-\infty}^x k'_1(t) f(t) dt + \tilde{g}_2(x) \int_x^{\infty} k'_2(t) dt.$$

Then $v(x)$ is a solution of (2.17). Since $|\tilde{g}_1(x)k'_1(t)| + |\tilde{g}_1'(x)k'_1(t)| \lesssim e^{-\nu(\eta)(x-t)}$ if $x > t$ and $|\tilde{g}_2(x)k'_2(t)| + |\tilde{g}_2'(x)k'_2(t)| \lesssim e^{\nu(\eta)(x-t)}$ if $x < t$, we have

$$|v_+(x)| + |\partial_x v_+(x)| \lesssim \int_{\mathbb{R}} e^{-\nu(\eta)|x-t|} |f(t)| dt.$$

Using Young's inequality, we have

$$\|v_+\|_{L_a^2(\mathbb{R})} + \|\partial_x v_+\|_{L_a^2(\mathbb{R})} \lesssim \|e^{-(\nu(\eta)-a)|x|}\|_{L^1(\mathbb{R})} \|f\|_{L_a^2(\mathbb{R})} \lesssim (\nu(\eta) - a)^{-1} \|f\|_{L_a^2(\mathbb{R})}.$$

Thus we can prove (2.37) in the same way as the proof of Lemma 2.6.

Now we prove (2). Let $v_- = T_3(f) + T_4(f)$. Obviously, v_- is a solution of (2.18) satisfying

$$|v_-(x)| + |\partial_x v_-(x)| \lesssim \int_{\mathbb{R}} e^{-\nu(\eta)|x-t|} |f(t)| dt.$$

Thus we can prove (2.38) in the same way as (2.37). This completes the proof of Lemma 2.8. \square

Using Lemmas 2.6, 2.7 and 2.8, we will investigate the spectrum $\sigma(\mathcal{L}(\eta))$ of $\mathcal{L}(\eta)$.

Lemma 2.9. *Let $a \in (0, 2)$ and η_* be a positive number satisfying $\nu(\eta_*) = a$.*

(1) *If $\eta \in (-\eta_*, \eta_*)$, then $\mathcal{L}(\eta)$ has no eigenvalue other than $\lambda(\pm\eta)$ and*

$$\sigma(\mathcal{L}(\eta)) = \{\lambda(\pm\eta_*)\} \cup \{ip(\xi + ia, \eta) \mid \xi \in \mathbb{R}\}.$$

(2) *If $\eta \in \mathbb{R} \setminus [-\eta_*, \eta_*]$, then $\sigma(\mathcal{L}(\eta)) = \{ip(\xi + ia, \eta) \mid \xi \in \mathbb{R}\}$.*

Proof of Lemma 2.9. The equation $\nu(\eta) = a$ has a unique positive root η_* because $\nu(\eta)$ is monotone increasing for $\eta \geq 0$, $\nu(0) = 0$ and $\nu(\infty) = \infty$.

Since $\lambda - \mathcal{L}(\eta)$ and $\lambda - \mathcal{L}_0(\eta)$ are invertible for large $\lambda > 0$ and $(\lambda - \mathcal{L}(\eta))^{-1} - (\lambda - \mathcal{L}_0(\eta))^{-1}$ is compact, it follows from the Weyl essential spectrum theorem that $\sigma(\mathcal{L}(\eta)) \setminus \sigma_p(\mathcal{L}(\eta)) = \{ip(\xi + ia) \mid \xi \in \mathbb{R}\}$.

Suppose that $\eta \in (-\eta_*, \eta_*)$ and that $\mathcal{L}(\eta)u = \lambda u$ for some $u \in H_a^3(\mathbb{R})$ and $\lambda \in \mathbb{C} \setminus \{\lambda(\pm\eta)\}$. Then

$$(2.39) \quad \int_{\mathbb{R}} u(x) \overline{g^*(x, \pm\eta)} dx = 0.$$

Indeed, it follows from Lemma 2.1 that

$$(\lambda - \lambda(\pm\eta)) \int_{\mathbb{R}} u(x) \overline{g^*(x, \pm\eta)} dx = \int_{\mathbb{R}} \{\lambda u(x) - (\mathcal{L}(\eta)u)(x)\} \overline{g^*(x, \pm\eta)} dx = 0.$$

Lemma 2.7 implies that there exists a solution $v \in H_a^4(\mathbb{R})$ of $u = \mathcal{M}_+(\eta)v$ satisfying $\int_{\mathbb{R}} v(x) \overline{g_M^*(x, \eta)} dx = 0$. By (2.32),

$$\begin{aligned} \mathcal{M}_+(\eta)(\mathcal{L}_M(\eta)v - \lambda v) &= (\mathcal{L}(\eta) - \lambda)\mathcal{M}_+(\eta)v \\ &= \mathcal{L}(\eta)u - \lambda u = 0. \end{aligned}$$

Since $\ker(\mathcal{M}_+(\eta)) = \{0\}$, it follows that $\mathcal{L}_M(\eta)v = \lambda v$. Using (2.33), we have

$$(2.40) \quad (\mathcal{L}_0(\eta) - \lambda)\mathcal{M}_-(\eta)v = \mathcal{M}_-(\eta)(\mathcal{L}_M(\eta)v - \lambda v) = 0,$$

whence $\mathcal{M}_-(\eta)v = 0$ because (2.40) implies that the support of $\mathcal{F}_x(\mathcal{M}_-(\eta)v)(\xi)$ is contained in $\{\xi \in \mathbb{R} \mid \xi^4 + 4\xi^2 + i\lambda\xi - \eta^2 = 0\}$. Lemma 2.6 implies there exists an $\alpha \in \mathbb{C}$ such that $v(x) = \alpha g_M(x, \eta)$ and hence it follows from (2.34) that

$$u(x) = \mathcal{M}_+(\eta)v = -2i\alpha\eta g(x, \eta).$$

By Lemma 2.1 and (2.39),

$$\int_{\mathbb{R}} u(x) \overline{g^*(x, \eta)} dx = -2i\alpha\eta = 0,$$

whence $u = 0$. Thus we prove (1).

Suppose $\eta \in \mathbb{R} \setminus [-\eta_*, \eta_*]$ and that $\mathcal{L}u = \lambda u$ for some $u \in H_a^3(\mathbb{R})$ and $\lambda \in \mathbb{C}$. Lemma 2.8 implies that there exists $v \in H_a^4(\mathbb{R})$ satisfying $u = \mathcal{M}_+(\eta)v$ and we can prove that $\mathcal{M}_-(\eta)v = 0$ in the same way as the proof of (1). Since $\mathcal{M}_-(\eta)$ has the bounded inverse, it follows that $v = 0$ and $u = \mathcal{M}_+(\eta)v = 0$. Thus we complete the proof. \square

3. SEMIGROUP ESTIMATES FOR THE LINEARIZED KP-II EQUATION

In this section, we will prove exponential decay estimates of solutions to (2.3). To begin with, we define a spectral projection to low frequency resonant modes. Let $P_0(\eta_0)$ be an operator defined by

$$P_0(\eta_0)f(x, y) = \frac{1}{2\pi} \sum_{k=1,2} \int_{-\eta_0}^{\eta_0} a_k(\eta) g_k(x, \eta) e^{iy\eta} d\eta,$$

$$\begin{aligned}
a_k(\eta) &= \int_{\mathbb{R}} \lim_{M \rightarrow \infty} \left(\int_{-M}^M f(x_1, y_1) e^{-iy_1 \eta} dy_1 \right) \overline{g_k^*(x_1, \eta)} dx_1 \\
&= \sqrt{2\pi} \int_{\mathbb{R}} (\mathcal{F}_y f)(x, \eta) \overline{g_k^*(x, \eta)} dx.
\end{aligned}$$

We will show that $P_0(\eta_0)$ is a spectral projection on $X = L^2(\mathbb{R}^2; e^{2ax} dx dy)$.

Lemma 3.1. *Let $a \in (0, 2)$ and η_1 be a positive constant satisfying $\nu(\eta_1) < a$. If $\eta_0 \in [-\eta_1, \eta_1]$, then*

- (1) $\|P_0(\eta_0)f\|_X + \|P_0(\eta_0)\partial_x f\|_X \leq C\|f\|_X$ for any $f \in X$, where C is a positive constant depending only on a and η_1 ,
- (2) $\|P_0(\eta_0)f\|_X + \|P_0(\eta_0)\partial_x f\|_X \leq C\|e^{ax}f\|_{L_x^1 L_y^2}$ for any $e^{ax}f \in L_x^1 L_y^2$, where C is a positive constant depending only on a and η_1 ,
- (3) $\mathcal{L}P_0(\eta_0)f = P(\eta_0)\mathcal{L}f$ for any $f \in D(\mathcal{L}) = \{u \mid u, \partial_x^3 u, \partial_x^{-1} \partial_y^2 u \in X\}$,
- (4) $P_0(\eta_0)^2 = P_0(\eta_0)$ on X ,
- (5) $e^{t\mathcal{L}}P_0(\eta_0) = P_0(\eta_0)e^{t\mathcal{L}}$ on X .

Proof. First, we will show (1). Since $C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ is dense in X , we may assume $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$. Let

$$f_k(x, y) = \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} a_k(\eta) g_k(x, \eta) e^{iy\eta} d\eta \quad \text{for } k = 1, 2.$$

By Plancherel's theorem,

$$\begin{aligned}
\|f_k(x, y)\|_{L_y^2} &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\eta_0}^{\eta_0} |a_k(\eta) g_k(x, \eta)|^2 d\eta \right)^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi}} \sup_{\eta \in [-\eta_0, \eta_0]} |g_k(x, \eta)| \left(\int_{-\eta_0}^{\eta_0} |a_k(\eta)|^2 d\eta \right)^{1/2}.
\end{aligned} \tag{3.1}$$

If $\nu(\eta_1) < a$, then it follows from the definition of g_k and g_k^* that there exists a positive constant C' such that for $\eta \in [-\eta_1, \eta_1]$ and $x \in \mathbb{R}$,

$$\begin{aligned}
|g_1(x, \eta)| &\leq C' \langle x \rangle e^{-2x_+} e^{\nu(\eta_1)x_-}, & |g_2(x, \eta)| &\leq C' e^{-2x_+} e^{\nu(\eta_1)x_-}, \\
|g_1^*(x, \eta)| &\leq C' e^{\nu(\eta_1)x_+} e^{-2x_-}, & |g_2^*(x, \eta)| &\leq C' \langle x \rangle e^{\nu(\eta_1)x_+} e^{-2x_-},
\end{aligned} \tag{3.2}$$

where $x_{\pm} = \max(\pm x, 0)$ and C' is a constant depending only on η_1 . Hence it follows from (3.1) and (5.4) that

$$\begin{aligned}
\|P_0(\eta_0)f\|_X &\leq \sum_{k=1,2} \left\| \|f_k\|_{L^2(\mathbb{R}_y)} \right\|_{L_a^2(\mathbb{R}_x)} \\
&\leq C_1 \left(\int_{-\eta_0}^{\eta_0} (|a_1(\eta)|^2 + |a_2(\eta)|^2) d\eta \right)^{1/2},
\end{aligned} \tag{3.3}$$

where C_1 is a constant depending only on a and η_1 . Using the Schwarz inequality and (3.2), we have for $\eta \in [-\eta_0, \eta_0]$,

$$|a_k(\eta)| \leq \sqrt{2\pi} \|(\mathcal{F}_y f)(x, \eta)\|_{L_a^2(\mathbb{R}_x)} \|g_k^*(x, \eta)\|_{L_{-a}^2(\mathbb{R}_x)} \leq C_2 \|(\mathcal{F}_y f)(x, \eta)\|_{L_a^2(\mathbb{R}_x)},$$

where C_2 is a constant depending only on a and η_1 . Hence it follows that for any $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$,

$$\begin{aligned} \|P_0(\eta)f\|_X &\leq C_1 C_2 \left(\int_{-\eta_0}^{\eta_0} \|\mathcal{F}_y f(x, \eta)\|_{L_a^2(\mathbb{R}_x)}^2 d\eta \right)^{1/2} \\ &= C_1 C_2 \|f\|_X. \end{aligned}$$

We can prove $\|P_0(\eta)\partial_x f\|_X \lesssim \|f\|_X$ in exactly the same way.

Next we will prove (2). Using Minkowski's inequality and applying (3.2) and Plancherel's theorem to the resulting equation, we have

$$\begin{aligned} \|a_k\|_{L^2(-\eta_0, \eta_0)} &\leq \sqrt{2\pi} \int_{\mathbb{R}} \|(\mathcal{F}f)(x, \cdot) \overline{g_k^*(x, \cdot)}\|_{L^2(-\eta_0, \eta_0)} dx \\ &\leq \sqrt{2\pi} \sup_{x \in \mathbb{R}, \eta \in [-\eta_0, \eta_0]} |e^{-ax} g_k^*(x, \eta)| \int_{\mathbb{R}} e^{ax} \|(\mathcal{F}f)(x, \cdot)\|_{L^2(-\eta_0, \eta_0)} dx \\ &\lesssim \|e^{ax} f\|_{L_x^1 L_y^2}. \end{aligned}$$

Substituting the above into (3.3), we have $\|P_0(\eta)f\|_X \lesssim \|e^{ax} f\|_{L_x^1 L_y^2}$. We can prove $\|P_0(\eta)\partial_x f\|_X \lesssim \|e^{ax} f\|_{L_x^1 L_y^2}$ in exactly the same way.

Since the potential of \mathcal{L} is independent of y , it suffices to show (3) for $f \in D(\mathcal{L}) \cap \tilde{X}$, where $\tilde{X} = \{f \in X \mid (\mathcal{F}_y f)(\cdot, \eta) = 0 \text{ a.e. } \eta \notin [-\eta_0, \eta_0]\}$. Since $\lambda(\pm\eta)$ are isolated eigenvalue of $\mathcal{L}(\eta)$ by Lemma 2.9, it follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned} P_0(\eta_0)f &= \frac{1}{(2\pi)^{3/2}i} \int_{-\eta_0}^{\eta_0} \int_{\Gamma} (\lambda - \mathcal{L}(\eta))^{-1} (\mathcal{F}_y f)(\cdot, \eta) e^{iy\eta} d\lambda d\eta \\ &= \frac{1}{2\pi i} \int_{\Gamma} (\lambda - \mathcal{L})^{-1} f d\lambda, \end{aligned}$$

where Γ is the boundary of a domain $D \supset \{\lambda(\pm\eta) \mid \eta \in [-\eta_0, \eta_0]\}$ satisfying $D \cap \{p(\eta + ia) \mid \eta \in \mathbb{R}\} = \emptyset$. Thus $P_0(\eta_0)$ equals to a spectral projection of $\mathcal{L}|_{\tilde{X}}$ defined by the Dunford integral and (3)–(5) can be obtained by a standard argument. We remark that $e^{t\mathcal{L}}$ is a C^0 -semigroup on X because $\mathcal{L}_0 := -\partial_x^3 + 4\partial_x - 3\partial_x^{-1}\partial_y^2$ is m -dissipative on X and $\mathcal{L} - \mathcal{L}_0$ is infinitesimally small with respect to \mathcal{L}_0 . Thus we complete the proof of Lemma 3.1. \square

Let $0 < \eta_1 \leq \eta_2 \leq \infty$ and $P_1(\eta_1, \eta_2)$ and $P_2(\eta_1, \eta_2)$ be projections defined by

$$\begin{aligned} P_1(\eta_1, \eta_2)u(x, y) &= \frac{1}{2\pi} \int_{\eta_1 \leq |\eta| \leq \eta_2} \int_{\mathbb{R}} u(x, y_1) e^{i\eta(y-y_1)} dy_1 d\eta, \\ P_2(\eta_1, \eta_2) &= P_1(0, \eta_2) - P_0(\eta_1). \end{aligned}$$

We remark that $P_2(\eta_1, \eta_2)$ is a projection onto non-resonant low frequency modes and that $\|P_2(\eta_1, \eta_2)e^{t\mathcal{L}}\|_{B(X)}$ decays exponentially as $t \rightarrow \infty$.

Proposition 3.2. *Let $a \in (0, 2)$ and η_1 be a positive number satisfying $\nu(\eta_1) < a$. Then there exist positive constants K and b such that for any $\eta_0 \in (0, \eta_1]$, $f \in X$ and $t \geq 0$,*

$$\|e^{t\mathcal{L}} P_2(\eta_0, \infty)f\|_X \leq K(\eta_0^{-1} e^{\Re \lambda(\eta_0)t} + e^{-bt}) \|f\|_X.$$

Corollary 3.3. *Let a and η_0 be as in Lemma 3.2. Then there exist positive constants K_1 and b such that for every $M \geq \eta_0$ and $f \in X$ and $t > 0$,*

$$(3.4) \quad \|e^{t\mathcal{L}} P_2(\eta_0, M) \partial_x f\|_X \leq K_1(1 + \eta_0^{-1} + t^{-1/2})e^{-bt} \|f\|_X,$$

$$(3.5) \quad \|e^{t\mathcal{L}} P_2(\eta_0, M) \partial_x f\|_X \leq K_1(1 + \eta_0^{-1} + t^{-3/4})e^{-bt} \|e^{ax} f\|_{L_x^1 L_y^2}.$$

To prove Proposition 3.2, we need decay estimates for the free semigroup $e^{t\mathcal{L}_0}$.

Lemma 3.4. *Let $a \in (0, 2)$. Then there exists a positive constant C such that for every $f \in C_0^\infty(\mathbb{R}^2)$ and $t > 0$,*

$$(3.6) \quad \|e^{t\mathcal{L}_0} f\|_X \leq C e^{-a(4-a^2)t} \|f\|_X,$$

$$(3.7) \quad \|e^{t\mathcal{L}_0} \partial_x f\|_X \leq C(1 + t^{-1/2}) e^{-a(4-a^2)t} \|f\|_X,$$

$$(3.8) \quad \|e^{t\mathcal{L}_0} \partial_x^{-1} \partial_y f\|_X \leq C t^{-1/2} e^{-a(4-a^2)t} \|f\|_X,$$

$$(3.9) \quad \|e^{t\mathcal{L}_0} \partial_x f\|_X \leq C(1 + t^{-3/4}) e^{-a(4-a^2)t} \|e^{ax} f\|_{L_y^2 L_x^1},$$

$$(3.10) \quad \|e^{t\mathcal{L}_0} f\|_X \leq C(t^{-1/2} + t^{-3/4}) e^{-a(4-a^2)t} \|e^{ax} f\|_{L^1(\mathbb{R}^2)}.$$

Proof. Let $u(t)$ be a solution to (2.12) satisfying the initial condition $u(0) = f$. Then

$$\hat{u}(t, \xi, \eta) = e^{itp(\xi, \eta)} \hat{f}(\xi, \eta), \quad p(\xi, \eta) = \xi^3 + 4\xi - \frac{3\eta^2}{\xi}.$$

It follows from Plancherel's theorem that for every $g \in X$,

$$(3.11) \quad \|g\|_X^2 = \int_{\mathbb{R}^2} e^{2ax} g(x, y)^2 dx dy = \int_{\mathbb{R}^2} |\hat{g}(\xi + ia, \eta)|^2 d\xi d\eta.$$

Making use of (3.11) and the fact that

$$(3.12) \quad \Im p(\xi + ia, \eta) = a(4 - a^2) + 3a\xi^2 + 3a\eta^2/(\xi^2 + a^2),$$

we have for $j \geq 0$,

$$\begin{aligned} \|\partial_x^j e^{t\mathcal{L}_0} f\|_X &\lesssim \left\| |\xi + ia|^j e^{-t\Im p(\xi + ia, \eta)} |\hat{f}(\xi + ia, \eta)| \right\|_{L^2} \\ &\lesssim e^{-a(4-a^2)t} \left(\sup_{\xi} (|\xi| + a)^j e^{-3at|\xi|^2} \right) \|\hat{f}(\cdot + ia, \cdot)\|_{L^2(\mathbb{R}^2)} \\ &\lesssim e^{-a(4-a^2)t} (1 + t^{-j/2}) \|f\|_X, \end{aligned}$$

and

$$\begin{aligned} \|(\partial_x^{-1} \partial_y)^j e^{t\mathcal{L}_0} f\|_X &\lesssim \left\| \frac{\eta^j}{(\xi^2 + a^2)^{j/2}} e^{-t\Im p(\xi + ia, \eta)} |\hat{f}(\xi + ia, \eta)| \right\|_{L^2} \\ &\lesssim e^{-a(4-a^2)t} \|\hat{f}(\cdot + ia, \cdot)\|_{L^2(\mathbb{R}^2)} \sup_{\xi, \eta} \left(\frac{\eta^2}{\xi^2 + a^2} \right)^{j/2} e^{-3a\eta^2 t/(\xi^2 + a^2)} \\ &\lesssim e^{-a(4-a^2)t} t^{-j/2} \|f\|_X. \end{aligned}$$

Similarly,

$$\begin{aligned} \|\partial_x^j e^{t\mathcal{L}_0} f\|_X &\lesssim \left\| \left\| |\xi + ia|^j e^{-t\Im p(\xi + ia, \eta)} |\hat{f}(\xi + ia, \eta)| \right\|_{L_\xi^2} \right\|_{L_\eta^2} \\ &\lesssim e^{-a(4-a^2)t} \left\| (|\xi| + a)^j e^{-3at|\xi|^2} \right\|_{L^2} \|\hat{f}(\cdot + ia, \cdot)\|_{L_\eta^2 L_\xi^\infty} \\ &\lesssim e^{-a(4-a^2)t} (1 + t^{-(2j+1)/4}) \|e^{ax} f\|_{L_y^2 L_x^1}, \end{aligned}$$

and

$$\begin{aligned} \|e^{t\mathcal{L}_0} f\|_X &\lesssim \left\| e^{-t\Im p(\xi + ia, \eta)} \right\|_{L_{\xi, \eta}^2} \left\| \hat{f}(\xi + ia, \eta) \right\|_{L_{\xi, \eta}^\infty} \\ &\lesssim e^{-a(4-a^2)t} \left\| e^{-3at\xi^2} \left\| e^{-3at\eta^2/(\xi^2 + a^2)} \right\|_{L_\eta^2} \right\|_{L_\xi^2} \|\hat{f}(\cdot + ia, \cdot)\|_{L_\eta^2 L_\xi^\infty} \\ &\lesssim e^{-a(4-a^2)t} (t^{-1/2} + t^{-3/4}) \|e^{ax} f\|_{L^1(\mathbb{R}^2)}. \end{aligned}$$

This completes the proof of Lemma 3.4. \square

Combining properties of the linearized Miura transformation and Lemma 3.4, we will prove linear decay estimates for non-resonant modes.

Lemma 3.5. *Let a and η_* be as in Lemma 2.9 and let $\eta_1 \in (0, \eta_*)$. Then there exists a positive constant K such that for every $t \geq 0$, $\eta_0 \in [-\eta_1, \eta_1]$ and $f \in C_0^\infty(\mathbb{R}^2)$,*

$$(3.13) \quad \|e^{t\mathcal{L}} P_2(\eta_0, \eta_0) f\|_X \leq K e^{-a(4-a^2)t} \|f\|_X.$$

Proof. Since $C_0^\infty(\mathbb{R})$ is dense in X , it suffices to prove (3.13) for $f \in C_0^\infty(\mathbb{R})$.

Let $u(t) = e^{t\mathcal{L}} P_2(\eta_0, \eta_0) f$. Since $P_0(\eta)$ is a spectral projection associated with \mathcal{L} (Lemma 3.1 (5)), we have $P_0(\eta_0)u(t) = 0$ for every $t \geq 0$. Let $u_\eta(t, x) = (\mathcal{F}_y u)(t, x, \eta)$. Then $u_\eta(t, \cdot) \in L_a^2(\mathbb{R})$ and $\int_{\mathbb{R}} u_\eta(t, x) \overline{g(x, \pm\eta)} dx = 0$ for *a.e.* $\eta \in [-\eta_0, \eta_0]$. Hence it follows from Lemma 2.7 that there exists $v_\eta(t, x) \in H_a^1(\mathbb{R})$ such that for $t \geq 0$ and *a.e.* $\eta \in [-\eta_0, \eta_0]$,

$$(3.14) \quad \begin{aligned} u_\eta(t, \cdot) &= \mathcal{M}_+(\eta) v_\eta(t, \cdot), \\ \int_{\mathbb{R}} v_\eta(t, x) \overline{g_M^*(x, \eta)} dx &= 0, \end{aligned}$$

$$(C_1 \|u_\eta(t)\|_{L_a^2(\mathbb{R})})^2 \leq \|v_\eta(t)\|_{H_a^1(\mathbb{R})}^2 + (|\eta| \|\partial_x^{-1} v_\eta(t)\|_{L_a^2(\mathbb{R})})^2 \leq (C_2 \|u_\eta(t)\|_{L_a^2(\mathbb{R})})^2,$$

where C_1 and C_2 are positive constants depending only on a and η_1 . Moreover,

$$v(t) = \frac{1}{\sqrt{2\pi}} \int_{-\eta_0}^{\eta_0} v_\eta(t, x, \eta) e^{iy\eta} d\eta$$

satisfies $u(t) = \nabla M_+(Q)v(t)$ for every $t \geq 0$. Hence it follows from Lemma 2.4 that $v(t)$ is a solution of (2.9) satisfying. Moreover, we have for $t \geq 0$,

$$(3.15) \quad \int_{\mathbb{R}} (\mathcal{F}_y v)(t, x, \eta) \overline{g_M^*(x, \eta)} e^{-iy\eta} dx = 0.$$

Integrating (3.14) over $[-\eta_0, \eta_0]$ and using Plancherel's theorem, we obtain

$$(3.16) \quad C_1 \|u(t)\|_X \leq \|v(t)\|_{X_M} \leq C_2 \|u(t)\|_X.$$

Let $\tilde{u}(t) = \nabla M_-(Q)v(t)$ and $\tilde{u}_\eta(t, x) = (\mathcal{F}_y \tilde{u})(t, x)$. Then $\tilde{u}_\eta(t) = \mathcal{M}_-(\eta)v_\eta(t)$ and it follows from Lemma 2.3 that $\tilde{u}(t)$ is a solution to (2.12). Using Lemma 2.6, we can prove that for $t \geq 0$,

$$(3.17) \quad C'_1 \|\tilde{u}(t)\|_X \leq \|v(t)\|_{X_M} \leq C'_2 \|\tilde{u}(t)\|_X,$$

in the same way as (3.16). Here C'_1 and C'_2 are positive constants depending only on a and η_1 . By Lemma 3.4,

$$(3.18) \quad \|\tilde{u}(t)\|_X \leq C \|\tilde{u}(0)\|_X e^{-a(4-a^2)t}.$$

Combining (3.16), (3.17) and (3.18), we obtain (3.13). Thus we complete the proof. \square

Lemma 3.6. *Let a and η_* be as in Lemma 2.9 and let $\eta_2 > \eta_*$. Then there exists a positive constant K such that for every $t \geq 0$ and $f \in C_0^\infty(\mathbb{R}^2)$,*

$$(3.19) \quad \|e^{t\mathcal{L}} P_1(\eta_2, \infty) f\|_X \leq K e^{-a(4-a^2)t} \|f\|_X.$$

Using Lemma 2.8 instead of Lemmas 2.6 and 2.7, we can prove Lemma 3.6 in exactly the same way as Lemma 3.5. Thus we omit the proof.

Middle frequency resonant modes are exponentially stable. We can obtain decay estimates of these modes by a direct computation.

Lemma 3.7. *Let a and η_* be as in Lemma 2.9. Let η_0 and η_1 be positive numbers satisfying $0 < \eta_0 < \eta_1 < \eta_*$. Then for every $f \in X$,*

$$\|e^{t\mathcal{L}}(P_0(\eta_1) - P_0(\eta_0))f\|_X \leq C(1 + \eta_0^{-1})e^{\Re\lambda(\eta_0)t} \|f\|_X,$$

where C is a constant depending only on a and η_1 .

Proof. Let $a_k(t, \eta) = \int_{\mathbb{R}} (\mathcal{F}_y u)(t, x, \eta) \overline{g_k^*(x, \eta)} e^{-iy\eta} dx$ for $k = 1, 2$ and let

$$E_a(t, \eta_0, \eta_1) = \int_{\eta_0 \leq |\eta| \leq \eta_1} (|a_1(t, \eta)|^2 + \eta^2 |a_2(t, \eta)|^2) d\eta.$$

Since $u(t)$ is a solution of (2.3), it follows from Lemma 2.2 that

$$(3.20) \quad \begin{aligned} \partial_t a_1(t, \eta) &= \int_{\mathbb{R}} \mathcal{L}(\eta) (\mathcal{F}_y u)(t, x, \eta) \overline{g_1^*(x, \eta)} dx \\ &= \Re\lambda(\eta) a_1 - \eta \Im\lambda(\eta) a_2, \end{aligned}$$

$$(3.21) \quad \begin{aligned} \partial_t a_2(t, \eta) &= \int_{\mathbb{R}} \mathcal{L}(\eta) (\mathcal{F}_y u)(t, x, \eta) \overline{g_2^*(x, \eta)} dx \\ &= \eta^{-1} \Im\lambda(\eta) a_1 + \Re\lambda(\eta) a_2, \end{aligned}$$

Using (3.20), (3.21) and the fact that $\Re\lambda(\eta)$ is even and monotone decreasing for $\eta \geq 0$,

$$\begin{aligned} \partial_t E_a(t, \eta_0, \eta_1) &= 2 \int_{\eta_0 \leq |\eta| \leq \eta_1} \Re\lambda(\eta) (|a_1(t, \eta)|^2 + \eta^2 |a_2(t, \eta)|^2) d\eta \\ &\leq 2\Re\lambda(\eta_0) E_a(t, \eta_0, \eta_1). \end{aligned}$$

Thus we have for $t \geq 0$,

$$(3.22) \quad E_a(t, \eta_0, \eta_1) \leq E_a(0, \eta_0, \eta_1) e^{2\Re\lambda(\eta_0)t}.$$

As in the proof of Lemma 3.1, we have

$$(3.23) \quad \|e^{t\mathcal{L}}(P_0(\eta_1) - P_0(\eta_0))f\|_X \leq C_1(1 + \eta_0^{-1}) E_a(t, \eta_0, \eta_1)^{1/2},$$

$$(3.24) \quad E_a(0, \eta_0, \eta_1)^{1/2} \leq C_2 \|f\|_X, \quad E_a(0, \eta_0, \eta_1)^{1/2} \leq C_2 \|e^{ax} f\|_{L_x^1 L_y^2},$$

where C_1 and C_2 are constants depending only on a and η_1 . By (3.22),

$$(3.25) \quad E_a(t, \eta_0, \eta_1) \leq e^{2\Re\lambda(\eta_0)t} E_a(0, \eta_0, \eta_1).$$

Combining (3.23)–(3.25), we obtain Lemma 3.7. Thus we complete the proof. \square

Now we are in position to prove Proposition 3.2.

Proof of Proposition 3.2. Let a_1, a_2, η_1 and η_2 be positive numbers satisfying $a_1 < \nu(\eta_1) < a < \nu(\eta_2) < a_2$. Note that $\eta_0 < \eta_1 < \eta_* < \eta_2$, where η_* is a root of $\nu(\eta) = a$. Since

$$P_2(\eta_0, \infty) = P_2(\eta_1, \eta_1) + P_0(\eta_1) - P_0(\eta_0) + P_1(\eta_1, \eta_2) + P_1(\eta_2, \infty),$$

it follows from Lemmas 3.5, 3.6 and 3.7 that

$$(3.26) \quad \|e^{t\mathcal{L}}(P_2(\eta_0, \infty) - P_1(\eta_1, \eta_2))f\|_X \leq (e^{-a(4-a_2)t} + (1 + \eta_0^{-1})e^{\Re\lambda(\eta_0)t})\|f\|_X.$$

In order to estimate $\|e^{t\mathcal{L}}P_1(\eta_1, \eta_2)f\|_X$, we will interpolate the decay estimate of $e^{t\mathcal{L}}P_1(\eta_1, \eta_2)$ in $L^2(\mathbb{R}^2; e^{2a_j x} dx dy)$ ($j = 1, 2$). Since $\nu(\eta_2) < a_2$, it follows from Lemmas 3.5 and 3.7 that for $t \geq 0$,

$$\begin{aligned} \|e^{a_2 x} e^{t\mathcal{L}}P_2(\eta_2, \eta_2)f\|_{L^2(\mathbb{R}^2)} &\lesssim e^{-a_2(4-a_2^2)t} \|e^{a_2 x} f\|_{L^2(\mathbb{R}^2)}, \\ \|e^{a_2 x} e^{t\mathcal{L}}(P_0(\eta_2) - P_0(\eta_1))f\|_{L^2(\mathbb{R}^2)} &\lesssim (e^{-a_2(4-a_2^2)t} + \eta_1^{-1}e^{\Re\lambda(\eta_1)t})\|e^{a_2 x} f\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Since $P_1(\eta_1, \eta_2) = P_2(\eta_1, \eta_2) + P_0(\eta_2) - P_1(\eta)$,

$$\|e^{a_2 x} e^{t\mathcal{L}}P_1(\eta_1, \eta_2)f\|_{L^2(\mathbb{R}^2)} \lesssim (e^{-a_2(4-a_2^2)t} + e^{\Re\lambda(\eta_1)t})\|e^{a_2 x} f\|_{L^2(\mathbb{R}^2)}.$$

On the other hand, Lemma 3.6 implies that

$$\|e^{a_1 x} e^{t\mathcal{L}}P_1(\eta_1, \eta_2)f\|_{L^2(\mathbb{R}^2)} \lesssim \|e^{a_1 x} f\|_{L^2(\mathbb{R}^2)}.$$

Hence it follows from the complex interpolation theorem that

$$(3.27) \quad \|e^{t\mathcal{L}}P_1(\eta_1, \eta_2)f\|_X \lesssim \left\{ e^{-a_1(4-a_1^2)t} + e^{-a_2(4-a_2^2)t} + (1 + \eta_1^{-1})e^{\Re\lambda(\eta_1)t} \right\} \|f\|_X.$$

By (3.26) and (3.27), we obtain

$$\begin{aligned} \|e^{t\mathcal{L}}(P_2(\eta_0, \infty))f\|_X &\lesssim \{e^{-a(4-a^2)t} + (1 + \eta_0^{-1})e^{\Re\lambda(\eta_0)t}\}\|f\|_X \\ &\quad + \{e^{-a_1(4-a_1^2)t} + e^{-a_2(4-a_2^2)t} + (1 + \eta_1^{-1})e^{\Re\lambda(\eta_1)t}\}\|f\|_X. \end{aligned}$$

Thus we complete the proof of Proposition 3.2. \square

Proof of Corollary 3.3. Without loss of generality, we may assume that $M = \infty$. By the variation of constants formula, we have for any $f \in X$,

$$(3.28) \quad \begin{aligned} e^{t\mathcal{L}}P_2(\eta_0, \infty)\partial_x f &= e^{t\mathcal{L}_0}P_2(\eta_0, \infty)\partial_x f \\ &\quad - 6 \int_0^t \partial_x e^{(t-s)\mathcal{L}_0} (\varphi e^{s\mathcal{L}}P_2(\eta_0, \infty)\partial_x f) ds. \end{aligned}$$

Let $t \in (0, 2]$. Applying Proposition 3.2 and Lemmas 3.1, 3.4 to (3.28),

$$\begin{aligned} \|e^{t\mathcal{L}}P_2(\eta_0, \infty)\partial_x f\|_X &\leq \|e^{t\mathcal{L}_0}P_2(\eta_0, \infty)\partial_x f\|_X \\ &\quad + 6 \int_0^t \left\| \partial_x e^{(t-s)\mathcal{L}_0} (\varphi e^{s\mathcal{L}}P_2(\eta_0, \infty)\partial_x f) \right\|_X \\ &\lesssim (1+t^{-1/2})\|f\|_X + \int_0^t (t-s)^{-1/2} \|e^{s\mathcal{L}}P_2(\eta_0, \infty)\partial_x f\|_{L^2}. \end{aligned}$$

By Gronwall's inequality, we have

$$(3.29) \quad \|e^{t\mathcal{L}}P_2(\eta_0, \infty)\partial_x f\|_X \leq Ct^{-1/2}\|f\|_X \quad \text{for } t \in (0, 2],$$

where C is a constant independent of $t \in (0, 2]$ and $f \in X$.

Let $t \geq 2$. Eq. (3.29) implies that $e^{\mathcal{L}}P_2(\eta_0, \infty)\partial_x$ is bounded on X . Applying Proposition 3.2 to $e^{t\mathcal{L}}P_2(\eta_0, \infty)\partial_x = e^{(t-1)\mathcal{L}}P_2(\eta_0, \infty)e^{\mathcal{L}}P_2(\eta_0, \infty)\partial_x$, we have for $t \geq 2$,

$$\|e^{t\mathcal{L}}P_2(\eta_0, \infty)\partial_x f\|_X \lesssim e^{-bt}\|f\|_X.$$

Combining the above with (3.29), we obtain (3.4).

Using (3.7) and (3.9) and Lemma 3.2, we can prove (3.5) in the same way as (3.4). \square

4. PRELIMINARIES

To begin with, we will introduce notation of Banach spaces which shall be used to analyze modulation equations. For an $\eta_0 > 0$, let Y and Z be closed subspaces of $L^2(\mathbb{R})$ defined by

$$Y = \mathcal{F}_\eta^{-1}Z \quad \text{and} \quad Z = \{f \in L^2(\mathbb{R}) \mid \text{supp } f \subset [-\eta_0, \eta_0]\},$$

and let $Y_1 = \mathcal{F}_\eta^{-1}Z_1$ and $Z_1 = \{f \in Z \mid \|f\|_{Z_1} := \|f\|_{L^\infty} < \infty\}$.

Remark 4.1. We have

$$(4.1) \quad \|f\|_{H^s} \leq (1 + \eta_0^2)^{s/2} \|f\|_{L^2} \quad \text{for any } s \geq 0 \text{ and } f \in Y,$$

since \hat{f} is 0 outside of $[-\eta_0, \eta_0]$. Especially, we have $\|f\|_{L^\infty} \lesssim \|f\|_{L^2}$ for any $f \in Y$.

Let \tilde{P}_1 be a projection defined by $\tilde{P}_1 f = \mathcal{F}_\eta^{-1} \mathbf{1}_{[-\eta_0, \eta_0]} \mathcal{F}_y f$, where $\mathbf{1}_{[-\eta_0, \eta_0]}(\eta) = 1$ for $\eta \in [-\eta_0, \eta_0]$ and $\mathbf{1}_{[-\eta_0, \eta_0]}(\eta) = 0$ for $\eta \notin [-\eta_0, \eta_0]$. Then $\|\tilde{P}_1 f\|_{Y_1} \leq (2\pi)^{-1/2} \|f\|_{L^1(\mathbb{R})}$ for any $f \in L^1(\mathbb{R})$. In particular, for any $f, g \in Y$,

$$(4.2) \quad \|\tilde{P}_1(fg)\|_{Y_1} \leq (2\pi)^{-1/2} \|fg\|_{L^1} \leq (2\pi)^{-1/2} \|f\|_Y \|g\|_Y.$$

In order to estimate modulation parameters $c(t, y)$ and $x(t, y)$, we will use a linear estimate for solutions to

$$(4.3) \quad \frac{\partial u}{\partial t} = A(t)u,$$

where $A(t) = A_0(D_y) + A_1(t, D_y)$, $u(t, y) = {}^t(u_1(t, y), u_2(t, y))$,

$$A_0(D_y) = \begin{pmatrix} a_{11}(D_y) & a_{12}(D_y)\partial_y \\ a_{21}(D_y)\partial_y & a_{22}(D_y) \end{pmatrix}, \quad A_1(t, D_y) = \begin{pmatrix} b_{11}(t, D_y) & b_{12}(t, D_y) \\ b_{21}(t, D_y) & b_{22}(t, D_y) \end{pmatrix},$$

and $a_{ij}(\eta)$ and $b_{ij}(t, \eta)$ are continuous in $\eta \in [-\eta_0, \eta_0]$ and $t \geq 0$. We denote by $U(t, s)f$ a solution of (4.3) satisfying $u(s, y) = f(y)$. Then we have the following.

Lemma 4.1. *Let $k \in \mathbb{Z}_{\geq 0}$, $\mu > 1/8$. Let $\delta_1, \delta_2, \kappa$ be positive constants. Suppose that $a_{ij}(\eta), b_{ij}(t, \eta)$ ($i, j = 1, 2$) satisfy*

$$(H) \quad \begin{aligned} &|a_{11}(\eta) + 3\eta^2| \leq \delta_1 \eta^2, \quad |a_{12}(\eta) - 8| \leq \delta_1 |\eta|, \\ &|a_{21}(\eta) - (2 + \mu\eta^2)| \leq \delta_1 \eta^2, \quad |a_{22}(\eta) + \eta^2| \leq \delta_1 \eta^2, \\ &|b_{ij}(t, \eta)| \leq \delta_2 e^{-\kappa t} \quad \text{for } j = 1, 2. \end{aligned}$$

If δ_1 is sufficiently small, then for every $t \geq s \geq 0$ and $f \in Y$,

$$(4.4) \quad \|\partial_y^k U(t, s)f\|_Y \leq C(1+t-s)^{-k/2} \|f\|_Y,$$

$$(4.5) \quad \|\partial_y^k U(t, s)f\|_Y \leq C(1+t-s)^{-(2k+1)/4} \|f\|_{Y_1},$$

where $C = C(\eta_0)$ is a constant satisfying $\limsup_{\eta_0 \downarrow 0} C(\eta_0) < \infty$.

Proof. We will prove Lemma 4.1 by the energy method. Let

$$(4.6) \quad \begin{aligned} \omega(\eta) &= \sqrt{16 + (8\mu - 1)\eta^2}, \\ A_*(\eta) &= \begin{pmatrix} -3\eta^2 & 8i\eta \\ i\eta(2 + \mu\eta^2) & -\eta^2 \end{pmatrix}, \quad \Pi_*(\eta) = \begin{pmatrix} 8i & 8i \\ \eta + i\omega(\eta) & \eta - i\omega(\eta) \end{pmatrix}. \end{aligned}$$

The matrix $A_*(\eta)$ has eigenvalues $\lambda_*^\pm = -2\eta^2 \pm i\eta\omega$ and $\Pi_*(\eta)^{-1}A_*(\eta)\Pi_*(\eta) = \text{diag}(\lambda_*^+(\eta), \lambda_*^-(\eta))$. By the assumption, there exist eigenvalues $\lambda^\pm(\eta)$ and an eigensystem $\Pi(\eta)$ of $A_0(\eta)$ satisfying for $\eta \in [-\eta_0, \eta_0]$,

$$|\lambda^\pm(\eta) - \lambda_*^\pm(\eta)| \lesssim \delta_1 \eta^2, \quad |\Pi(\eta) - \Pi_*(\eta)| \lesssim \delta_1.$$

Let $\Lambda(\eta) = \text{diag}(\lambda^+(\eta), \lambda^-(\eta))$, $B(t, \eta) = \Pi(\eta)^{-1}A_1(t, \eta)\Pi(\eta)$ and

$$\mathbf{e}(t, \eta) = \begin{pmatrix} e_+(t, \eta) \\ e_-(t, \eta) \end{pmatrix} = \Pi(\eta)^{-1}(\mathcal{F}_y u)(t, \eta).$$

Then (4.3) can be rewritten as

$$\partial_t \mathbf{e}(t, \eta) = (\Lambda(\eta) + B(t, \eta))\mathbf{e}(t, \eta).$$

Differentiating the energy function $e(t, \eta) := |e_+(t, \eta)|^2 + |e_-(t, \eta)|^2$ with respect to t , we have

$$(4.7) \quad \begin{aligned} \partial_t e(t, \eta) &= 2 \sum_{\pm} \Re \lambda_{\pm}(\eta) |e_{\pm}(t, \eta)|^2 + 2 \Re \langle B(t, \eta) \mathbf{e}(t, \eta), \mathbf{e}(t, \eta) \rangle_{\mathbb{C}^2} \\ &\leq (-4 + O(\delta_1)) \eta^2 e(t, \eta) + C \delta_2 e^{-\kappa t} e(t, \eta), \end{aligned}$$

where C is a positive constant and $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ is the standard inner product on \mathbb{C}^2 . By Gronwall's inequality, there exists a positive constant c_3 such that

$$(4.8) \quad e(t, \eta) \leq c_3 e(s, \eta) e^{(-4 + O(\delta_1))\eta^2(t-s)} \quad \text{for } t \geq s \geq 0.$$

Since $\|\partial_y^k u(t)\|_Y^2 \simeq \int_{|\eta| \leq \eta_0} \eta^{2k} e(t, \eta) d\eta$,

$$\begin{aligned} \|\partial_y^k u(t)\|_Y^2 &\lesssim \left(\sup_{|\eta| \leq \eta_0} \eta^{2k} e^{-4\eta^2(t-s)} \right) \int_{|\eta| \leq \eta_0} e(s, \eta) d\eta \\ &\lesssim \langle t-s \rangle^{-k} \|u(s)\|_Y^2 \quad \text{for } t \geq s \geq 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|\partial_y^k u(t)\|_Y^2 &\lesssim \sup_{|\eta| \leq \eta_0} e(s, \eta) \int_{|\eta| \leq \eta_0} \eta^{2k} e^{(-4+O(\delta_1))\eta^2(t-s)} d\eta \\ &\lesssim \langle t-s \rangle^{-(2k+1)/2} \|u(s)\|_{Y_1}^2 \quad \text{for } t \geq s \geq 0. \end{aligned}$$

Thus we complete the proof \square

Let $A_* = \mathcal{F}^{-1} A_*(\eta) \mathcal{F}$, where $A_*(\eta)$ is a matrix defined by (4.6). For a specific choice of μ , we can express the semigroup e^{tA_*} by using the kernel $H_t(y) = (4\pi t)^{-1/2} e^{-y^2/4t}$.

Lemma 4.2. *Let $\mu = 1/8$ and $(f_1, f_2) \in Y \times Y$. Then*

$$(4.9) \quad e^{tA_*} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} k_{11}(t, \cdot) * f_1 + k_{12}(t, \cdot) * f_2 \\ k_{21}(t, \cdot) * f_1 + k_{22}(t, \cdot) * f_2 \end{pmatrix},$$

where

$$\begin{aligned} k_{11}(t, y) &= \left(\frac{1}{2} + \frac{1}{8} \partial_y \right) H_{2t}(y+4t) + \left(\frac{1}{2} - \frac{1}{8} \partial_y \right) H_{2t}(y-4t), \\ k_{12}(t, y) &= H_{2t}(y+4t) - H_{2t}(y-4t), \\ k_{21}(t, y) &= \left(\frac{1}{4} - \frac{1}{64} \partial_y^2 \right) (H_{2t}(y+4t) - H_{2t}(y-4t)), \\ k_{22}(t, y) &= \left(\frac{1}{2} - \frac{1}{8} \partial_y \right) H_{2t}(y+4t) + \left(\frac{1}{2} + \frac{1}{8} \partial_y \right) H_{2t}(y-4t). \end{aligned}$$

Moreover, for every $k \in \mathbb{Z}_{\geq 0}$, there exists a positive constant C such that

$$\|\partial_y^k e^{tA_*}\|_{B(Y, Y)} \leq C \langle t \rangle^{-k/2}, \quad \|\partial_y^k e^{tA_*}\|_{B(Y_1, Y)} \leq C \langle t \rangle^{-(2k+1)/4}.$$

Proof. In view of the proof of Lemma 4.1,

$$e^{tA_*}(\eta) = e^{-2t\eta^2} \begin{pmatrix} \cos 4t\eta - \frac{\eta}{4} \sin 4t\eta & 2i \sin 4t\eta \\ \left(\frac{i\eta^2}{32} + \frac{i}{2} \right) \sin 4t\eta & \cos 4t\eta + \frac{\eta}{4} \sin 4t\eta \end{pmatrix},$$

provided $\mu = 1/8$. Taking the inverse Fourier transform of the above, we obtain (4.9). The decay estimates follow immediately from (4.9) and the fact that $\|\partial_y^k H_t\|_{B(Y, Y)} \lesssim \langle t \rangle^{-k/2}$ and $\|\partial_y^k H_t\|_{B(Y_1, Y)} \lesssim \langle t \rangle^{-(2k+1)/4}$. Thus we complete the proof. \square

Using (4.9) and the fact that $\|\partial_y^k H_{2t} * f\|_Y \lesssim \langle t \rangle^{-(2k+1)/4} \|f\|_{Y_1}$ for $t \geq 0$ and $k \in \mathbb{Z}_{\geq 0}$, we can obtain the first order asymptotics of $e^{tA_*}(f_1, f_2)$ as $t \rightarrow \infty$.

Corollary 4.3. *Let μ and A_* be as in Lemma 4.2. Then there exists a positive constant C such that for every $(f_1, f_2) \in Y_1 \times Y_1$,*

$$\left\| e^{tA_*} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - e^{4t\partial_y} H_{2t} * \begin{pmatrix} 2f_+ \\ f_+ \end{pmatrix} - e^{-4t\partial_y} H_{2t} * \begin{pmatrix} 2f_- \\ -f_- \end{pmatrix} \right\|_Y \leq C \langle t \rangle^{-3/4} \sum_{i=1,2} \|f_i\|_{Y_1},$$

where $f_+ = \frac{1}{4}f_1 + \frac{1}{2}f_2$ and $f_- = \frac{1}{4}f_1 - \frac{1}{2}f_2$.

To estimate inhomogeneous terms of modulation equations, we will use the following.

Claim 4.1. *Let α and β be positive constants and $\gamma = \min\{\alpha, \beta, \alpha + \beta - 1\}$. If (α, β) satisfies i) $\alpha \neq 1$ and $\beta \neq 1$ or ii) $\alpha > \beta = 1$ or (iii) $\beta > \alpha = 1$, then there exists a positive constant C such that*

$$\int_0^t \langle t-s \rangle^{-\alpha} \langle s \rangle^{-\beta} ds \leq C \langle t \rangle^{-\gamma}.$$

If $\alpha < 1$ and $\beta = 1$ or $\alpha = 1$ and $\beta < 1$, then there exists a $C > 0$ such that

$$\int_0^t \langle t-s \rangle^{-\alpha} \langle s \rangle^{-\beta} ds \leq C \langle t \rangle^{-\min(\alpha, \beta)} \log \langle t \rangle.$$

5. DECOMPOSITION OF THE PERTURBED LINE SOLITON

In this section, we will decompose a solution around a line soliton solution $\varphi(x - 4t)$ into a sum of a modulating line soliton and a non-resonant dispersive part plus a small wave which is caused by amplitude changes of the line soliton:

$$(5.1) \quad u(t, x, y) = \varphi_{c(t, y)}(z) - \psi_{c(t, y), L}(z + 4t) + v(t, z, y), \quad z = x - x(t, y).$$

The modulation parameters $c(t_0, y_0)$ and $x(t_0, y_0)$ denote the maximum height and the phase shift of the modulating line soliton $\varphi_{c(t, y)}(x - x(t, y))$ along the line $y = y_0$ at the time $t = t_0$, and $\psi_{c, L}$ is an auxiliary function such that

$$(5.2) \quad \int_{\mathbb{R}} \psi_{c, L}(x) dx = \int_{\mathbb{R}} (\varphi_c(x) - \varphi(x)) dx.$$

More precisely,

$$\psi_{c, L}(x) = 2(\sqrt{2c} - 2)\psi(x + L),$$

where $L > 0$ is a large constant to be fixed later and $\psi(x)$ is a nonnegative function such that $\psi(x) = 0$ if $|x| \geq 1$ and that $\int_{\mathbb{R}} \psi(x) dx = 1$. Since a localized solution to KP-type equations satisfies $\int_{\mathbb{R}} u(t, x, y) dx = 0$ for any $y \in \mathbb{R}$ and $t > 0$ (see [27]), it is natural to expect small perturbations appear in the rear of the solitary wave if the solitary wave is amplified.

To fix the decomposition (5.1), we impose that $v(t, z, y)$ is symplectically orthogonal to low frequency resonant modes. More precisely, we impose the constraint that for $k = 1, 2$,

$$(5.3) \quad \lim_{M \rightarrow \infty} \int_{-M}^M \int_{\mathbb{R}} v(t, z, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy = 0 \quad \text{in } L^2(-\eta_0, \eta_0),$$

where $g_1^*(x, \eta, c) = c g_1^*(\sqrt{c/2}x, \eta)$ and $g_2^*(x, \eta, c) = \frac{c}{2} g_2^*(\sqrt{c/2}x, \eta)$.

We will show that the decomposition (5.1) with (5.3) is well defined if u is close to a modulating line soliton in the exponentially weighted space X . It is expected that $\|c(t, \cdot) - 2\|_{L^\infty}$ remains small as long as (5.1) persists.

Now let us introduce functional to prove the existence of the representation (5.1) that satisfies the orthogonality condition (5.3). For $v \in X$ and $\gamma, \tilde{c} \in Y$ and $L \geq 0$, let $c(y) = 2 + \tilde{c}(y)$ and

$$\begin{aligned} F_k[u, \tilde{c}, \gamma, L](\eta) := & \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) \lim_{M \rightarrow \infty} \int_{-M}^M \int_{\mathbb{R}} \{u(x, y) + \varphi(x) - \varphi_{c(y)}(x - \gamma(y)) \\ & + \psi_{c(y), L}(x - \gamma(y))\} \overline{g_k^*(x - \gamma(y), \eta, c(y))} e^{-iy\eta} dx dy. \end{aligned}$$

To begin with, we will show that $F = (F_1, F_2)$ is a mapping from $X \times Y \times Y \times \mathbb{R}$ into $Z \times Z$.

Lemma 5.1. *Let $a \in (0, 2)$, $u \in X = L^2(\mathbb{R}^2; e^{2ax} dx dy)$, $\tilde{c}, \gamma \in Y$ and $L \geq 0$. Then there exists a $\delta > 0$ such that if $\|\tilde{c}\|_Y + \|\gamma\|_Y \leq \delta$, then $F_k[u, \tilde{c}, \gamma, L] \in Z$ for $k = 1, 2$. Moreover, if $u \in X_1 := L^1(\mathbb{R}_y; L_a^2(\mathbb{R}_x))$ and $\tilde{c}, \gamma \in Y_1$, then $F_k[u, \tilde{c}, \gamma, L] \in Z_1$ for $k = 1, 2$.*

Proof. Let $u \in C_0^\infty(\mathbb{R}^2)$ and

$$\begin{aligned}\Phi_1(x, y) &= \varphi_{c(y)}(x - \gamma(y)) - \varphi(x) - \psi_{c(y), L}(x - \gamma(y)), \\ \Phi_{1,0}(x, y) &= \partial_c \varphi(x) \tilde{c}(y) - \varphi'(x) \gamma(y) - \psi_{c(y), L}(x), \\ \Phi_2(x, y) &= \Phi_1(x, y) - \Phi_{1,0}(x, y), \quad \Psi(x, y) = \overline{g_k^*(x - \gamma(y), \eta, c(y))} - \overline{g_k^*(x, \eta)}.\end{aligned}$$

Then

$$\int_{\mathbb{R}^2} \{u(x, y) - \Phi_1(x, y)\} \overline{g_k^*(x - \gamma(y), \eta, c(y))} e^{-iy\eta} dx dy = \sum_{j=1}^4 I_j(\eta).$$

where

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^2} u(x, y) \overline{g_k^*(x, \eta)} e^{-iy\eta} dx dy, \\ I_2 &= - \int_{\mathbb{R}^2} \Phi_{1,0}(x, y) \overline{g_k^*(x, \eta)} e^{-iy\eta} dx dy, \\ I_3 &= - \int_{\mathbb{R}^2} \Phi_2(x, y) \overline{g_k^*(x, \eta)} e^{-iy\eta} dx dy, \\ I_4 &= \int_{\mathbb{R}^2} \{u(x, y) - \Phi_1(x, y)\} \Psi(x, y) e^{-iy\eta} dx dy.\end{aligned}$$

By Claim 2.1,

$$(5.4) \quad \sup_{c \in [2-\delta, 2+\delta]} \sup_{\eta \in [-\eta_0, \eta_0]} \|\partial_c^j \partial_x^k g_k^*(\cdot, \eta, c)\|_{L_{-a}^2(\mathbb{R})} < \infty \quad \text{for } j, k \geq 0,$$

and it follows from Plancherel's theorem and (5.4) that

$$\int_{-\eta_0}^{\eta_0} |I_1(\eta)|^2 d\eta \lesssim \|u\|_X^2, \quad \int_{-\eta_0}^{\eta_0} |I_2(\eta)|^2 d\eta \lesssim \|\tilde{c}\|_Y^2 + \|\gamma\|_Y^2.$$

Since $\sup_y (|\tilde{c}(y)| + |\gamma(y)|) \lesssim \|\tilde{c}\|_Y + \|\gamma\|_Y$, we have

$$\begin{aligned}\|\Phi_1\|_X + \|\Phi_{1,0}\|_X &\leq C(\|\tilde{c}\|_Y + \|\gamma\|_Y), \\ \|e^{ax} \Phi_2(x, y)\|_{L^1(\mathbb{R}^2)} &\leq C(\|\tilde{c}\|_Y + \|\gamma\|_Y)^2, \\ \|e^{-ax} \Psi(x, y)\|_{L^2(\mathbb{R}^2)} &\leq C(\|\tilde{c}\|_Y + \|\gamma\|_Y),\end{aligned}$$

where C is a positive constant depending only on δ .

Combining the above, we obtain

$$\sup_{-\eta_0 \leq \eta \leq \eta_0} (|I_3(\eta)| + |I_4(\eta)|) \lesssim \|u\|_X (\|\tilde{c}\|_Y + \|\gamma\|_Y) + (\|\tilde{c}\|_Y + \|\gamma\|_Y)^2.$$

Since $C_0^\infty(\mathbb{R}^2)$ is dense in X , it follows that for any $u \in X$,

$$\mathbf{1}_{[-\eta_0, \eta_0]}(I_1 + I_2) \in Z, \quad \mathbf{1}_{[-\eta_0, \eta_0]}(I_3 + I_4) \in Z_1 \subset Z.$$

Suppose $u \in X_1$ and $\tilde{c}, \gamma \in Y_1$. Noting that $\sqrt{2c} - 2 = \tilde{c}/2 + O(\tilde{c}^2) \in Y_1$, we have

$$\sup_{[-\eta_0, \eta_0]} |I_1(\eta)| \lesssim \|u\|_{X_1}, \quad \sup_{[-\eta_0, \eta_0]} |I_2(\eta)| \lesssim \|\tilde{c}\|_{Y_1} + \|\gamma\|_{Y_1},$$

and $\mathbf{1}_{[-\eta_0, \eta_0]} \sum_{1 \leq i \leq 4} I_i \in Z_1$. Thus we complete the proof. \square

Lemma 5.2. *Let $a \in (0, 2)$. There exist positive constants δ_0, δ_1, L_0 and C such that if $\|u\|_X < \delta_0$ and $L \geq L_0$, then there exists a unique (\tilde{c}, γ) with $c = 2 + \tilde{c}$ satisfying*

$$(5.5) \quad \|\tilde{c}\|_Y + \|\gamma\|_Y < \delta_1,$$

$$(5.6) \quad F_1[u, \tilde{c}, \gamma, L] = F_2[u, \tilde{c}, \gamma, L] = 0.$$

Moreover, the mapping $\{u \in X \mid \|u\|_X < \delta_0\} \ni u \mapsto (\tilde{c}, \gamma) =: \Phi(u)$ is C^1 .

Proof. Clearly, we have $(F_1, F_2) \in C^1(X \times Y \times Y \times \mathbb{R}; Z \times Z)$ and for $\tilde{c}, \tilde{\gamma} \in Y$,

$$D_{(c, \gamma)}(F_1, F_2)(0, 0, 0, L) \begin{pmatrix} \tilde{c} \\ \tilde{\gamma} \end{pmatrix} = \sqrt{2\pi} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \mathcal{F}_y \tilde{c} \\ \mathcal{F}_y \tilde{\gamma} \end{pmatrix},$$

where

$$\begin{aligned} f_{11} &= - \int_{\mathbb{R}} (\partial_c \varphi(x) - \psi(x+L)) g_1^*(x, \eta) dx, & f_{12} &= \int_{\mathbb{R}} \varphi'(x) g_1^*(x, \eta) dx, \\ f_{21} &= - \int_{\mathbb{R}} (\partial_c \varphi(x) - \psi(x+L)) g_2^*(x, \eta) dx, & f_{22} &= \int_{\mathbb{R}} \varphi'(x) g_2^*(x, \eta) dx. \end{aligned}$$

By Claims 2.1 and A.1 in Appendix A,

$$\begin{aligned} f_{11} &= -1 + O(\eta_0^2) + O(e^{-aL}), & f_{12} &= O(\eta_0^2), \\ f_{21} &= -\frac{1}{2} + O(\eta_0^2) + O(e^{-aL}), & f_{22} &= -2 + O(\eta_0^2), \end{aligned}$$

and $D_{(c, \gamma)}(F_1, F_2)(0, 0, 0, L) \in B(Y \times Y, Z \times Z)$ has a bounded inverse if δ_0 and e^{-aL} are sufficiently small. Hence it follows from the implicit function theorem that for any u satisfying $\|u\|_X < \delta_0$, there exists a unique $(c, \gamma) \in Y \times Y$ satisfying (5.5) and (5.6). Moreover, the mapping $(\tilde{c}, \gamma) = \Phi(u)$ is C^1 . \square

Remark 5.1. In Lemma 5.2, we can replace X by a Banach space X_2 whose norm is

$$\|u\|_{X_2} = \left(\int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}} \frac{|\hat{u}(\xi + ia, \eta)|^2}{(1 + \xi^2)^3} d\xi d\eta \right)^{1/2}.$$

Suppose $u(t, x, y)$ is a solution of (2.1) satisfying $u(0, x, y) = \varphi(x) + v_0(x, y)$ and $v_0 \in X \cap H^1(\mathbb{R})$. Then for any $T > 0$,

$$(5.7) \quad \tilde{v}(t, x, y) := u(t, x + 4t, y) - \varphi(x) \in C([0, \infty); X),$$

(see Proposition E.1 in Appendix E). Combining (5.7) and the fact that $\tilde{v}(t)$ is a solution of (E.1), we have $\partial_t P_1(0, \eta_0)u \in C([0, \infty); X_2)$ and

$$(5.8) \quad P_1(0, \eta_0)\tilde{v}(t) \in C^1([0, \infty); X_2).$$

If $\sup_{t \in [0, T)} \|\tilde{v}(t)\|_{X_2}$ is sufficiently small for a $T > 0$, then there exists $(\tilde{c}(t), \tilde{x}(t)) := \Phi(\tilde{v}(t))$ satisfying (5.3) for $t \in [0, T)$, where $c(t, y) = \tilde{c}(t, y) + 2$ and $x(t, y) = 4t + \tilde{x}(t, y)$ and v and z are defined by (5.1). That is, the decomposition (5.1)

satisfying (5.3) exists on $[0, T]$ if $\|v_0\|_{X_2} \lesssim \|v_0\|_X$ is sufficiently small. Since $X_2 \ni u \mapsto \Phi(u) \in Y \times Y$ is C^1 , it follows from (5.8) that

$$(\tilde{c}(t), \tilde{x}(t)) = \Phi(\tilde{v}(t)) \in C([0, T]; Y \times Y) \cap C^1((0, T); Y \times Y).$$

We use the following lemma to decompose initial data around the line soliton.

Lemma 5.3. *Let $a \in (0, 2)$. There exist positive constants δ_2, δ_3 and L'_0 such that if $\|u\|_{X_1} < \delta_2$ and $L \geq L'_0$, then there exists a unique (\tilde{c}, γ) with $c = 2 + \tilde{c}$ satisfying $\|\tilde{c}\|_{Y_1} + \|\gamma\|_{Y_1} < \delta_3$ and (5.6).*

Lemma 5.3 can be proved in exactly the same as Lemma 5.2.

We provide a continuation principle that ensures the existence of (5.1) as long as $\|v(t)\|_X$ and $\|\tilde{c}(t)\|_Y$ remain small.

Proposition 5.4. *Let a, δ_0 and L be the same as in Lemma 5.2 and let $u(t)$ be a solution of (2.1) such that $u(t, x, y) - \varphi(x - 4t) \in C([0, \infty); X \cap L^2(\mathbb{R}^2))$. Then there exists a constant $\delta_2 > 0$ such that if (5.1) and (5.3) hold for $t \in [0, T]$ and $v(t, z, y), \tilde{c}(t, y) := c(t, y) - 2$ and $\tilde{x}(t, y) := x(t, y) - 4t$ satisfy*

$$(5.9) \quad (\tilde{c}, \tilde{x}) \in C([0, T]; Y \times Y) \cap C^1((0, T); Y \times Y),$$

$$(5.10) \quad \sup_{t \in [0, T)} \|v(t)\|_X \leq \frac{\delta_0}{2}, \quad \sup_{t \in [0, T)} \|\tilde{c}(t)\|_Y < \delta_2, \quad \sup_{t \in [0, T)} \|\tilde{x}(t)\|_Y < \infty,$$

then either $T = \infty$ or T is not the maximal time of the decomposition (5.1) satisfying (5.3), (5.9) and (5.10).

Proof. Suppose $T < \infty$. Let $\tau \in (0, T)$, $\bar{x}(t, y) = x(t, y)$ for $t \in [0, T - \tau]$ and $\bar{x}(t, y) = x(T - \tau, y) + 4(t + \tau - T)$ for $t \geq T - \tau$. Let $u_1(t, x, y) = u(t, x + \bar{x}(t, y), y) - \varphi(x)$. Then

$$\begin{aligned} \sup_{t \in [0, T - \tau]} \|u_1(t)\|_X &\leq \sup_{t \in [0, T)} (\|v(t)\|_X + \|\varphi_{c(t, y)} - \varphi\|_X + \|\tilde{\psi}_{c(t, y)}\|_X) \\ &\leq \frac{\delta_0}{2} + C_1 \sup_{t \in [0, T)} \|\tilde{c}(t)\|_Y \leq \frac{\delta_0}{2} + C_1 \delta_2, \end{aligned}$$

where $\tilde{\psi}_c(x) = \psi_{c, L}(x + 4t)$ and C_1 is a constant that does not depend on τ . Since $Y \subset L^\infty(\mathbb{R})$, it follows from the assumption that $C_2 := \sup_{\tau \in (0, T)} \sup_y e^{-a\bar{x}(\tau, y)} < \infty$. Thus for $t \in [T - \tau, T)$,

$$\begin{aligned} \|u_1(t)\|_X &\leq \|u_1(T - \tau)\|_X + \left\| e^{-a\bar{x}(T - \tau, y)} \{\tilde{v}(t) - \tilde{v}(T - \tau)\} \right\|_X \\ &\leq \frac{\delta_0}{2} + C_1 \delta_2 + C_2 \|\tilde{v}(t) - \tilde{v}(T - \tau)\|_X. \end{aligned}$$

Now we choose δ_2 and τ so that

$$\delta_2 < \min \{\delta_1, \delta_0/(4C_1)\}, \quad \sup_{t_1, t_2 \in [T - \tau, T + \tau]} \|\tilde{v}(t_2) - \tilde{v}(t_1)\|_X < \delta_0/(4C_2).$$

Then we have $\sup_{t \in [0, T + \tau]} \|u_1(t)\|_X < \delta_0$ and it follows from Lemma 5.2 and Remark 5.1 that there exists a unique

$$(\tilde{c}_1(t), \tilde{x}_1(t)) \in C([T - \tau, T + \tau]; Y \times Y) \cap C^1((T - \tau, T + \tau); Y \times Y)$$

satisfying $\sup_{t \in (T-\tau, T+\tau)} (\|\tilde{c}_1(t)\|_Y + \|\tilde{x}_1(t)\|_Y) < \delta_1$ and

$$(5.11) \quad u(t, x + \tilde{x}(t, y), y) = \varphi_{c_1(t, y)}(z_1) - \tilde{\psi}_{c_1(t, y)}(z_1) + v_1(t, z_1, y),$$

$$(5.12) \quad \lim_{M \rightarrow \infty} \int_{-M}^M \int_{\mathbb{R}} v_1(t, z_1, y) \overline{g_k^*(z_1, \eta, c_1(t, y))} e^{-iy\eta} dz_1 dy = 0 \quad \text{in } L^2(-\eta_0, \eta_0)$$

for $k = 1$ and 2 , where $c_1(t, y) = 2 + \tilde{c}_1(t, y)$ and $z_1 = x - x_1(t, y)$. By the local uniqueness of the decomposition, we have for $t \in [T - \tau, T]$,

$$(5.13) \quad \tilde{c}(t) = \tilde{c}_1(t), \quad \tilde{x}(t) = \tilde{x}(T - \tau) + \tilde{x}_1(t), \quad v(t, x, y) = v_1(t, x, y).$$

Let us define $\tilde{c}(t)$ and $\tilde{x}(t)$ by (5.13) and $v(t)$ by (5.1) for $t \in [T, T + \tau]$. Then $(\tilde{c}, \tilde{x}) \in C([0, T + \tau]; Y \times Y) \cap C^1((0, T + \tau), Y \times Y)$ and (5.11) and (5.12) imply that $v(t)$ satisfy (5.3) for $t \in [0, T + \tau]$. Thus we prove that T is not maximal. This completes the proof of Proposition 5.4. \square

6. MODULATION EQUATIONS

In this section, we will derive a system of PDEs which describe the motion of $c(t, y)$ and $x(t, y)$. Substituting the ansatz (5.1) into (2.1), we obtain

$$(6.1) \quad \partial_t v = \mathcal{L}_c v + \ell + \partial_z(N_1 + N_2) + N_3,$$

where $\mathcal{L}_c v = -\partial_z(\partial_z^2 - 2c + 6\varphi_c)v - 3\partial_z^{-1}\partial_y^2$, $\ell = \ell_1 + \ell_2$, $\ell_k = \ell_{k1} + \ell_{k2} + \ell_{k3}$ ($k = 1, 2$) and

$$\begin{aligned} \ell_{11} &= (x_t - 2c - 3(x_y)^2)\varphi'_c - (c_t - 6c_y x_y)\partial_c \varphi_c, \quad \ell_{12} = 3x_{yy}\varphi_c, \\ \ell_{13} &= 3c_{yy} \int_z^\infty \partial_c \varphi_c(z_1) dz_1 + 3(c_y)^2 \int_z^\infty \partial_c^2 \varphi_c(z_1) dz_1, \\ \ell_{21} &= (c_t - 6c_y x_y)\partial_c \tilde{\psi}_c - (x_t - 4 - 3(x_y)^2)\tilde{\psi}'_c, \\ \ell_{22} &= \partial_z^3 \tilde{\psi}_c - 3\partial_z(\tilde{\psi}_c^2) + 6\partial_z(\varphi_c \tilde{\psi}_c) - 3x_{yy}\tilde{\psi}_c, \\ \ell_{23} &= -3c_{yy} \int_z^\infty \partial_c \tilde{\psi}_c(z_1) dz_1 - 3(c_y)^2 \int_z^\infty \partial_c^2 \tilde{\psi}_c(z_1) dz_1, \\ N_1 &= -3v^2, \quad N_2 = \{x_t - 2c - 3(x_y)^2\}v + 6\tilde{\psi}_c v, \\ N_3 &= 6x_y \partial_y v + 3x_{yy}v = 6\partial_y(x_y v) - 3x_{yy}v. \end{aligned}$$

Here we abbreviate $c(t, y)$ as c and $x(t, y)$ as x .

First, we will derive modulation equations of $c(t, y)$ and $x(t, y)$ from the orthogonality condition (5.3) assuming that $v_0 \in X \cap H^3(\mathbb{R}^2)$ and $\partial_x^{-1}v_0 \in H^2(\mathbb{R}^2)$. If $v_0 \in H^3(\mathbb{R}^2)$ and $\partial_x^{-1}v_0 \in H^2(\mathbb{R}^2)$, then it follows from [29] that $\tilde{v}(t) \in H^3(\mathbb{R}^2) \subset C(\mathbb{R}; H^3(\mathbb{R}^2))$ and $\partial_x^{-1}\tilde{v}(t) \in C(\mathbb{R}; H^2(\mathbb{R}^2))$. Moreover, Proposition E.1 implies that $\tilde{v}(t) \in C([0, \infty); X)$. If $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$ are sufficiently small, then the decomposition (5.1) satisfying (5.3) and (5.9) exists for $t \in [0, T]$ by Lemma 5.2, Remark 5.1 and Proposition 5.4. Since $Y \subset H^4(\mathbb{R})$,

$$v(t, z, y) - \tilde{v}(t, z + \tilde{x}(t, y), y) = \varphi(z + x(t, y)) - \varphi_{c(t, y)}(z) + \tilde{\psi}_{c(t, y)}(z) \in H^3(\mathbb{R}^2),$$

and we easily see that $v(t) \in C([0, T]; X \cap H^3(\mathbb{R}^2))$. Using

$$\int_{\mathbb{R}} (v(t, z, y) - \tilde{v}(t, z + \tilde{x}(t, y), y)) dz = 0,$$

by (5.2) and its integrand decays exponentially as $z \rightarrow \pm\infty$, we have

$$(\partial_z^{-1}v)(t, z, y) = - \int_z^\infty v(t, z_1, y) dz_1 \in X \cap H^2(\mathbb{R}^2).$$

By Proposition 5.4 and Remark 5.1, the mapping

$$t \mapsto \int_{\mathbb{R}^2} v(t, z, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \in Z$$

is C^1 for $t \in [0, T]$ if we have (5.9) and (5.10). Differentiating (5.3) with respect to t and substituting (6.1) into the resulting equation, we have in $L^2(-\eta_0, \eta_0)$

$$(6.2) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} v(t, z, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \\ &= \int_{\mathbb{R}^2} \ell g_k^*(z, \eta, c(t, y)) e^{-iy\eta} dz dy + \sum_{j=1}^5 II_k^j(t, \eta) = 0, \end{aligned}$$

where

$$\begin{aligned} II_k^1 &= \int_{\mathbb{R}^2} v(t, z, y) \mathcal{L}_{c(t, y)}^* (\overline{g_k^*(t, z, c(t, y))} e^{iy\eta}) dz dy, \\ II_k^2 &= - \int_{\mathbb{R}^2} N_1 \overline{\partial_z g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_k^3 &= \int_{\mathbb{R}^2} N_3 \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \\ &\quad + 6 \int_{\mathbb{R}^2} v(t, z, y) c_y(t, y) x_y(t, y) \overline{\partial_c g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_k^4 &= \int_{\mathbb{R}^2} v(t, z, y) (c_t - 6c_y x_y)(t, y) \overline{\partial_c g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\ II_k^5 &= - \int_{\mathbb{R}^2} N_2 \overline{\partial_z g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy. \end{aligned}$$

Next, we will show the second equation of (6.2) for $t \in [0, T]$ and $v(t) \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^\infty([0, T]; X)$ assuming that $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$ are sufficiently small. Let $\{v_{0n}\}_{n=1}^\infty$ be a sequence such that

$$v_{0n} \in H^3(\mathbb{R}^2) \cap X, \quad \partial_x^{-1} v_{0n} \in H^2(\mathbb{R}^2), \quad \lim_{n \rightarrow \infty} (\|v_{0n} - v_0\|_X + \|v_{0n} - v_0\|_{L^2(\mathbb{R}^2)}) = 0,$$

and let $u_n(t)$ be a solution of (2.1) satisfying $u_n(0, x, y) = \varphi(x) + v_{0n}(x, y)$ and $\tilde{v}_n(t, x, y) = u_n(t, x, y) - \varphi(x)$. Since $\sup_{t \in [0, T]} \|\tilde{v}_n(t) - \tilde{v}(t)\|_{L^2(\mathbb{R}^2)} \rightarrow 0$ as $n \rightarrow \infty$ by [29] and $\sup_n \sup_{t \in [0, T]} \|\tilde{v}_n(t)\|_X < \infty$ by (E.4) in Appendix E, we have $\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\tilde{v}_n(t) - \tilde{v}(t)\|_{L^2(\mathbb{R}^2; e^{az} dz)} = 0$ for any $t \geq 0$. If η_0 is so small that $a/2 > \nu_0$, we can replace the weight function e^{2az} by e^{az} in Lemma 5.2, Remark 5.1 and Proposition 5.4 and see that there exist $v_n(t)$, $c_n(t)$ and $x_n(t)$ satisfying for

$t \in [0, T]$,

$$u_n(t, x, y) = \varphi_{c_n(t, y)}(z) - \psi_{c_n(t, y), L}(z + 4t) + v_n(t, z, y), \quad z = x - x_n(t, y).$$

$$\lim_{M \rightarrow \infty} \int_{-M}^M \int_{\mathbb{R}} v_n(t, z, y) \overline{g_k^*(z, \eta, c_n(t, y))} e^{-iy\eta} dz dy = 0 \quad \text{in } L^2(-\eta_0, \eta_0),$$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|v_n(t) - v(t)\|_{L^2(\mathbb{R}^2; e^{ax} dz dy)} = 0,$$

$$\lim_{n \rightarrow \infty} \|(c_n(t) - c(t), x_n(t) - x(t))\|_{C^1([0, T]; Y \times Y)} = 0.$$

Thus we can obtain the second equation of (6.2) on $[0, T]$ for $v_0 \in X \cap L^2(\mathbb{R}^2)$ by passing to the limit $n \rightarrow \infty$.

The modulation PDEs of $c(t, y)$ and $x(t, y)$ can be obtained by computing the inverse Fourier transform of (6.2) in the η -variable. The leading term of

$$\int_{\mathbb{R}^2} \ell_1 \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy$$

is $\sqrt{2\pi} \mathcal{F}_y G_k(t, \eta)$, where

$$(6.3) \quad G_k(t, y) = \int_{\mathbb{R}} \ell_1 \overline{g_k^*(z, 0, c(t, y))} dz.$$

Using the asymptotic formula of $g_k^*(z, \eta)$, we can see that G_k has the following expression.

Lemma 6.1. *Let $\mu_1 = \frac{1}{2} - \frac{\pi^2}{12}$ and $\mu_2 = \frac{\pi^2}{32} - \frac{3}{16}$. Then*

$$\begin{aligned} G_1 &= 16x_{yy} \left(\frac{c}{2}\right)^{3/2} - 2(c_t - 6c_y x_y) \left(\frac{c}{2}\right)^{1/2} + 6c_{yy} - \frac{3}{c}(c_y)^2, \\ G_2 &= -2(x_t - 2c - 3(x_y)^2) \left(\frac{c}{2}\right)^2 + 6x_{yy} \left(\frac{c}{2}\right)^{3/2} - \frac{1}{2}(c_t - 6c_y x_y) \left(\frac{c}{2}\right)^{1/2} \\ &\quad + \mu_1 c_{yy} + \mu_2 (c_y)^2 \left(\frac{c}{2}\right)^{-1}. \end{aligned}$$

The proof is given in Appendix A. We remark that (G_1, G_2) are the dominant part of diffusion wave equations for c and x .

Next, we will expand

$$\int_{\mathbb{R}} \ell_1 \left(\overline{g_k^*(z, \eta, c(t, y))} - \overline{g_k^*(z, 0, c(t, y))} \right) e^{-iy\eta} dz dy$$

in $c(t, y)$ and $x(t, y)$ up to the second order. In order to express the coefficients of c_t , x_t , c_{yy} and x_{yy} , let us introduce the operators S_k^j ($j, k = 1, 2$). For $q_c(z) = \varphi_c(z)$, $\varphi'_c(z)$, $\partial_c \varphi_c(z)$ and $\partial_z^{-1} \partial_c^m \varphi_c(z) = -\int_z^\infty \partial_c^m \varphi_c(z_1) dz_1$ ($m \geq 1$), let

$$\begin{aligned} S_k^1[q_c](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) q_2(z) \overline{g_{k1}^*(z, \eta, 2)} e^{i(y-y_1)\eta} dy_1 dz d\eta, \\ S_k^2[q_c](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \tilde{c}(t, y_1) \overline{g_{k2}^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta, \end{aligned}$$

where

$$\begin{aligned} g_{k1}^*(z, \eta, c) &= \frac{g_k^*(z, \eta, c) - g_k^*(z, 0, c)}{\eta^2}, \quad \delta q_c(z) = \frac{q_c(z) - q_2(z)}{c - 2}, \\ g_{k2}^*(z, \eta, c) &= g_{k1}^*(z, \eta, 2) \delta q_c(z) + \frac{g_{k1}^*(z, \eta, c) - g_{k1}^*(z, \eta, 2)}{c - 2} q_c(z). \end{aligned}$$

We have $S_k^1 \in B(Y)$ and S_k^1 are independent of $c(t, y)$ whereas $\|S_k^2\|_{B(Y, Y_1)} \lesssim \|\tilde{c}\|_Y$. See Claims B.1 and B.2 in Appendix B. Using S_k^j ($j, k = 1, 2$), we have

$$(6.4) \quad \begin{aligned} & \frac{1}{\sqrt{2\pi}} \tilde{P}_1 \mathcal{F}_\eta^{-1} \left(\int_{\mathbb{R}} \ell_1 \left(\overline{g_k^*(z, \eta, c(t, y))} - g_k^*(z, 0, c(t, y)) \right) e^{-iy\eta} dz dy \right) \\ &= - \sum_{j=1,2} \partial_y^2 \left(S_k^j[\varphi_c'](x_t - 2c - 3(x_y)^2) - S_k^j[\partial_c \varphi_c](c_t - 6c_y x_y) \right) - \partial_y^2 (R_k^1 + R_k^2), \\ & R_k^1 = 3S_k^1[\varphi_c](x_{yy}) - 3S_k^1[\partial_z^{-1} \partial_c \varphi_c](c_{yy}), \\ & R_k^2 = 3S_k^2[\varphi_c](x_{yy}) - 3S_k^2[\partial_z^{-1} \partial_c \varphi_c](c_{yy}) - 3 \sum_{j=1,2} S_k^j[\partial_z^{-1} \partial_c^2 \varphi_c](c_y^2). \end{aligned}$$

We rewrite the linear term R_k^1 as

$$\begin{pmatrix} R_1^1 \\ R_2^1 \end{pmatrix} = \tilde{S}_0 \begin{pmatrix} c_{yy} \\ x_{yy} \end{pmatrix}, \quad \tilde{S}_0 = 3 \begin{pmatrix} -S_1^1[\partial_z^{-1} \partial_c \varphi_c] & S_1^1[\varphi_c] \\ -S_2^1[\partial_z^{-1} \partial_c \varphi_c] & S_2^1[\varphi_c] \end{pmatrix}.$$

Next we deal with $\int_{\mathbb{R}} \ell_2 \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy$. Let $S_k^3[p]$ and $S_k^4[p]$ be operators defined by

$$\begin{aligned} S_k^3[p](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) p(z + 4t + L) \overline{g_k^*(z, \eta)} e^{i(y-y_1)\eta} dy_1 dz d\eta, \\ S_k^4[p](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \tilde{c}(t, y_1) p(z + 4t + L) \\ &\quad \times \overline{g_{k3}^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta, \end{aligned}$$

where

$$g_{k3}^*(z, \eta, c) = \frac{g_k^*(z, \eta, c) - g_k^*(z, \eta)}{c - 2}.$$

By the definition of $\tilde{\psi}_c$,

$$(6.5) \quad \begin{aligned} & \frac{\mathbf{1}_{[-\eta_0, \eta_0]}(\eta)}{\sqrt{2\pi}} \int_{\mathbb{R}} \ell_{21} \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \\ &= \mathcal{F}_y \left\{ (S_k^3[\psi] + S_k^4[\psi]) (\sqrt{2/c}(c_t - 6c_y x_y)) \right\} (t, \eta) \\ &\quad - 2\sqrt{2} \mathcal{F}_y \left\{ (S_k^3[\psi'] + S_k^4[\psi']) ((\sqrt{c} - \sqrt{2})(x_t - 4 - 3(x_y)^2)) \right\} (t, \eta). \end{aligned}$$

The operator norms of $S_k^j[\psi]$, $S_k^j[\psi']$ ($j = 3, 4, k = 1, 2$) decay exponentially because $g_k^*(z, \eta)$ and $g_k^*(z, \eta, c)$ are exponentially localized as $z \rightarrow -\infty$ and $\psi \in C_0^\infty(\mathbb{R})$. See Claims B.3 and B.4 in Appendix B.

Next we decompose $(2\pi)^{-1} \int_{\mathbb{R}^2} (\ell_{22} + \ell_{23}) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy$ into a linear part and a nonlinear part with respect to \tilde{c} and \tilde{x} . The linear part can be written as

$$(6.6) \quad \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \ell_{2,lin}(t, z, y_1) \overline{g_k^*(z, \eta)} e^{i(y-y_1)\eta} dy_1 dz d\eta =: \tilde{a}_k(t, D_y) \tilde{c},$$

where

$$\ell_{2,lin}(t, z, y) = \tilde{c}(t, y) \partial_z \{ \partial_z^2 + 6\varphi(z) \} \psi(z + 4t + L) - 3c_{yy}(t, y) \int_z^\infty \psi(z_1 + 4t + L) dz_1,$$

$$\begin{aligned}
(6.7) \quad \tilde{a}_k(t, \eta) = & \left[\int_{\mathbb{R}} \{ \psi'''(z + 4t + L) + 6(\varphi(z)\psi(z + 4t + L))_z \} \overline{g_k^*(z, \eta)} dz \right. \\
& \left. + 3\eta^2 \int_{\mathbb{R}} \left(\int_z^\infty \psi(z_1 + 4t + L) dz_1 \right) \overline{g_k^*(z, \eta)} dz \right] \mathbf{1}_{[-\eta_0, \eta_0]}(\eta),
\end{aligned}$$

and the nonlinear part is

$$\begin{aligned}
(6.8) \quad R_k^3(t, y) := & \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}} (\ell_{22} + \ell_{23}) \overline{g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy_1 d\eta \\
& - \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}} \ell_{2,lin} \overline{g_k^*(z, \eta)} e^{i(y-y_1)\eta} dz dy_1 d\eta,
\end{aligned}$$

Next, we deal with II_k^j ($j = 1, \dots, 5$) in (6.2). Let

$$\begin{aligned}
II_{k1}^3 &= -3 \int_{\mathbb{R}^2} v(t, z, y) x_{yy}(t, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \\
II_{k2}^3 &= 6 \int_{\mathbb{R}^2} v(t, z, y) x_y(t, y) \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy
\end{aligned}$$

so that $II_k^3 = II_{k1}^3 + i\eta II_{k2}^3$. Let

$$\begin{aligned}
(6.9) \quad R_k^4(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \{ II_k^1(t, \eta) + II_k^2(t, \eta) + II_{k1}^3(t, \eta) \} e^{iy\eta} d\eta, \\
R_k^5(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} II_{k2}^3(t, \eta) e^{iy\eta} d\eta.
\end{aligned}$$

Let S_k^5 and S_k^6 be operators defined by

$$\begin{aligned}
S_k^5 f(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v(t, z, y_1) f(y_1) \overline{\partial_c g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy_1 d\eta, \\
S_k^6 f(t, y) &= -\frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} v(t, z, y_1) f(y_1) \overline{\partial_z g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy_1 d\eta
\end{aligned}$$

so that

$$\begin{aligned}
(6.10) \quad \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) II_k^4(t, \eta) &= \sqrt{2\pi} \mathcal{F}_y(S_k^5(c_t - 6c_y x_y)), \\
\mathbf{1}_{[-\eta_0, \eta_0]}(\eta) II_k^5(t, \eta) &= \sqrt{2\pi} \mathcal{F}_y \{ S_k^6(x_t - 2c - 3(x_y)^2) + R_k^6 \},
\end{aligned}$$

where

$$R_k^6 = -\frac{3}{\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \psi_{c(t, y_1), L}(z + 4t) v(t, z, y_1) \overline{\partial_z g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta.$$

Now we are in position to translate (6.2) into a PDE form. Using (6.3)–(6.5) and (6.6)–(6.10), we can translate (6.2) into

$$\begin{aligned}
(6.11) \quad & \tilde{P}_1 \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} - \left(\partial_y^2(\tilde{S}_1 + \tilde{S}_2) - \tilde{S}_3 - \tilde{S}_4 - \tilde{S}_5 \right) \begin{pmatrix} c_t - 6c_y x_y \\ x_t - 2c - 3(x_y)^2 \end{pmatrix} \\
& + \tilde{\mathcal{A}}_1(t) \begin{pmatrix} \tilde{c} \\ x \end{pmatrix} - \partial_y^2 R_1 + \tilde{R}^1 + \partial_y \tilde{R}^2 = 0,
\end{aligned}$$

where

$$\begin{aligned}\tilde{S}_j &= \begin{pmatrix} -S_1^j[\partial_c \varphi_c] & S_1^j[\varphi'_c] \\ -S_2^j[\partial_c \varphi_c] & S_2^j[\varphi'_c] \end{pmatrix} \quad \text{for } j = 1, 2, \quad \tilde{S}_3 = \begin{pmatrix} S_1^3[\psi] & 0 \\ S_2^3[\psi] & 0 \end{pmatrix}, \\ \tilde{S}_4 &= \begin{pmatrix} S_1^3[\psi]((\sqrt{2/c}-1)\cdot) + S_1^4[\psi](\sqrt{2/c}\cdot) & -2(S_1^3[\psi'] + S_1^4[\psi'])((\sqrt{2c}-2)\cdot) \\ S_1^4[\psi]((\sqrt{2/c}-1)\cdot) + S_2^4[\psi](\sqrt{2/c}\cdot) & -2(S_2^3[\psi'] + S_2^4[\psi'])((\sqrt{2c}-2)\cdot) \end{pmatrix}, \\ \tilde{S}_5 &= \begin{pmatrix} S_1^5 & S_1^6 \\ S_2^5 & S_2^6 \end{pmatrix}, \quad \tilde{\mathcal{A}}_1(t) = \begin{pmatrix} \tilde{a}_1(t, D_y) & 0 \\ \tilde{a}_2(t, D_y) & 0 \end{pmatrix}, \\ \tilde{R}^1 &= R^3 + R^4 + R^6 - \tilde{S}_4 \begin{pmatrix} 0 \\ 2\tilde{c} \end{pmatrix}, \quad \tilde{R}^2 = R^5 - \partial_y R^2,\end{aligned}$$

and $R^j = {}^t(R_1^j, R_2^j)$ for $1 \leq j \leq 6$. In G_1 , the nonlinear terms $6(c/2)^{1/2}c_yx_y$ and $16x_{yy}\{((c/2)^{3/2}-1)\}$ are critical because they are expected to decay like t^{-1} as $t \rightarrow \infty$. To translate these nonlinearity into a divergence form, we will make use of the following change of variables. Let

$$\begin{aligned}(6.12) \quad \tilde{x}(t, \cdot) &= x(t, \cdot) - 4t, \quad b(t, \cdot) = \frac{1}{3}\tilde{P}_1 \left\{ \sqrt{2c}(t, \cdot)^{3/2} - 4 \right\}, \\ \mathcal{C}_1 &= \frac{1}{2}\tilde{P}_1 \{c(t, \cdot)^2 - 4\} \tilde{P}_1, \\ \tilde{\mathcal{C}}_1 &= \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C}_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & 0 \\ \frac{1}{2} & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 6 & 16 \\ \mu_1 & 6 \end{pmatrix}.\end{aligned}$$

We remark that $b \simeq \tilde{c} = c - 2$ if c is close to 2 (see Claim D.6). By (6.12), we have $b_t = \tilde{P}_1(c/2)^{1/2}c_t$, $b_y = \tilde{P}_1(c/2)^{1/2}c_y$ and it follows from Lemma 6.1 that

$$(6.13) \quad \tilde{P}_1 \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = - (B_1 + \tilde{\mathcal{C}}_1) \tilde{P}_1 \begin{pmatrix} b_t - 6(bx_y)_y \\ x_t - 2c - 3(x_y)^2 \end{pmatrix} + B_2 \begin{pmatrix} b_{yy} \\ x_{yy} \end{pmatrix} + \tilde{P}_1 R^7,$$

where $R^7 = {}^t(R_1^7, R_2^7)$ and

$$\begin{aligned}(6.14) \quad R_1^7 &= \left\{ 4\sqrt{2}c^{3/2} - 16 - 12b \right\} x_{yy} - 6(b_{yy} - c_{yy}) \\ &\quad - 6(2b_y - (2c)^{1/2}c_y)x_y - 3c^{-1}(c_y)^2, \\ R_2^7 &= 6 \left\{ \left(\frac{c}{2} \right)^{3/2} - 1 \right\} x_{yy} + 3 \left(\frac{c}{2} \right)^{1/2} c_y x_y - 3(bx_y)_y \\ &\quad - \mu_1(b_{yy} - c_{yy}) + \mu_2 \frac{2}{c} (c_y)^2.\end{aligned}$$

Let $\mathcal{C}_2 = \tilde{P}_1 \left\{ \left(\frac{c(t, \cdot)}{2} \right)^{1/2} - 1 \right\} \tilde{P}_1$, $\tilde{\mathcal{C}}_2 = \begin{pmatrix} \mathcal{C}_2 & 0 \\ 0 & 0 \end{pmatrix}$, $\tilde{S}_j = \tilde{S}_j(I + \tilde{\mathcal{C}}_2)^{-1}$ for $1 \leq j \leq 5$ and

$$(6.15) \quad B_3 = B_1 + \tilde{\mathcal{C}}_1 + \partial_y^2(\tilde{S}_1 + \tilde{S}_2) - \tilde{S}_3 - \tilde{S}_4 - \tilde{S}_5.$$

Note that $I + \tilde{\mathcal{C}}_2$ is invertible as long as $c(t, \cdot)$ remains small in Y and that B_3 is a bounded operator on $Y \times Y$ depending on \tilde{c} and v . Substituting (6.13) into (6.11), we have

$$\begin{aligned}& B_3 \tilde{P}_1 \begin{pmatrix} b_t - 6(bx_y)_y \\ x_t - 2c - 3(x_y)^2 \end{pmatrix} \\ &= \left\{ (B_2 - \partial_y^2 \tilde{S}_0) \partial_y^2 + \tilde{\mathcal{A}}_1(t) \right\} \begin{pmatrix} b \\ x \end{pmatrix} + \tilde{P}_1 R^7 + \tilde{R}^1 + \partial_y \tilde{R}^2 + \tilde{R}^3,\end{aligned}$$

where $\tilde{R}^3 = R^8 + R^9 + R^{10}$ and

$$\begin{aligned} R^8 &= 6(B_3 - B_1 - \tilde{C}_1) \begin{pmatrix} (I + \mathcal{C}_2)(c_y x_y) - (bx_y)_y \\ 0 \end{pmatrix}, \\ R^9 &= \partial_y^2 \tilde{S}_0 \begin{pmatrix} b_{yy} - c_{yy} \\ 0 \end{pmatrix}, \quad R^{10} = \tilde{\mathcal{A}}_1(t) \begin{pmatrix} \tilde{c} - b \\ 0 \end{pmatrix}. \end{aligned}$$

Let

$$\begin{aligned} \mathbb{M}_1(T) &= \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^1 (1+t)^{(2k+1)/4} (\|\partial_y^k \tilde{c}(t, \cdot)\|_{L^2} + \|\partial_y^{k+1} x(t, \cdot)\|_{L^2}) \right. \\ &\quad \left. + (1+t)(\|\partial_y^2 \tilde{c}(t, \cdot)\|_{L^2} + \|\partial_y^3 x(t, \cdot)\|_{L^2}) \right\}, \\ \mathbb{M}_2(T) &= (1+t)^{3/4} \|v(t, \cdot)\|_X, \quad \mathbb{M}_3(T) = \sup_{0 \leq t \leq T} \|v(t, \cdot)\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Then we have the following.

Proposition 6.2. *There exists a $\delta_3 > 0$ such that if $\mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + e^{-aL} < \delta_3$ for a $T \geq 0$, then*

$$(6.16) \quad \begin{pmatrix} b_t \\ \tilde{x}_t \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} b \\ \tilde{x} \end{pmatrix} + \sum_{i=1}^8 \mathcal{N}_i$$

where $B_4 = B_1 + \partial_y^2 \tilde{S}_1 - \tilde{S}_3 = B_3|_{\tilde{c}=0, v=0}$,

$$\begin{aligned} \mathcal{A}(t) &= B_4^{-1} \left\{ (B_2 - \partial_y^2 \tilde{S}_0) \partial_y^2 + \tilde{\mathcal{A}}_1(t) \right\} + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \\ \mathcal{N}_1 &= \tilde{P}_1 \begin{pmatrix} 6(b\tilde{x}_y)_y \\ 2(\tilde{c} - b) + 3(\tilde{x}_y)^2 \end{pmatrix}, \quad \mathcal{N}_2 = B_3^{-1} \tilde{P}_1 R^7, \\ \mathcal{N}_3 &= B_3^{-1} \tilde{R}^1, \quad \mathcal{N}_4 = B_3^{-1} \partial_y \tilde{R}^2, \quad \mathcal{N}_5 = B_3^{-1} \tilde{R}^3, \\ \mathcal{N}_6 &= (B_3^{-1} - B_4^{-1}) \tilde{\mathcal{A}}_1(t) \begin{pmatrix} b \\ x \end{pmatrix}, \quad \mathcal{N}_7 = (B_3^{-1} - B_4^{-1}) (B_2 - \partial_y^2 \tilde{S}_0) \begin{pmatrix} b_{yy} \\ 0 \end{pmatrix}, \\ \mathcal{N}_8 &= (B_3^{-1} - B_4^{-1}) (B_2 - \partial_y^2 \tilde{S}_0) \begin{pmatrix} 0 \\ x_{yy} \end{pmatrix}. \end{aligned}$$

Proof. Proposition 5.4 implies that the (5.1) persists on $[0, T]$ if δ_3 is sufficiently small. Moreover Claims 6.1–6.3 below imply that B_3 , B_4 and $I + \tilde{\mathcal{C}}_k$ are invertible if $\|\tilde{c}(t)\|_Y$, $\|v(t)\|_X$, η_0 and e^{-aL} are sufficiently small. Thus we have (6.16). \square

Claim 6.1. *There exist positive constants δ and C such that if $\mathbb{M}_1(T) \leq \delta$, then for $s \in [0, T]$ and $k = 1, 2$,*

$$(6.17) \quad \|\tilde{\mathcal{C}}_k\|_{B(Y)} \leq C \mathbb{M}_1(T) \langle s \rangle^{-1/2},$$

$$(6.18) \quad \|\tilde{\mathcal{C}}_k\|_{B(Y, Y_1)} \leq C \mathbb{M}_1(T) \langle s \rangle^{-1/4},$$

$$\|(I + \tilde{\mathcal{C}}_k)^{-1}\|_{B(Y)} + \|(I + \tilde{\mathcal{C}}_k)^{-1}\|_{B(Y_1)} \leq C.$$

Claim 6.1 immediately follows from Claim B.6 in Appendix B and the fact that $Y_1 \subset Y$ and $\|\tilde{\mathcal{C}}_k\|_{B(Y_1)} \lesssim \|\mathcal{C}_k\|_{B(Y, Y_1)}$.

Claim 6.2. *There exist positive constants δ and C such that if $\mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0^2 + e^{-aL} \leq \delta$, then*

$$\|B_3^{-1}\|_{B(Y)} \leq C \quad \text{and} \quad \|B_3^{-1}\|_{B(Y_1)} \leq C \quad \text{for } s \in [0, T].$$

Claim 6.3. *There exist positive constants C and δ such that if $\eta_0^2 + e^{-aL} \leq \delta$, then $\|B_4^{-1}\|_{B(Y)} + \|B_4^{-1}\|_{B(Y_1)} \leq C$.*

The proof of Claims 6.2 and 6.3 will be given in Appendix C.

7. À PRIORI ESTIMATES FOR $c(t, y)$ AND $x_y(t, y)$

In this section, we will estimate $\mathbb{M}_1(T)$ assuming that $\mathbb{M}_i(T)$ ($1 \leq i \leq 3$), η_0 and e^{-aL} are sufficiently small.

Lemma 7.1. *There exist positive constants δ_4 and C such that if $\mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + e^{-aL} \leq \delta_4$, then*

$$(7.1) \quad \mathbb{M}_1(T) \leq C\|v_0\|_{X_1} + C(\mathbb{M}_1(T) + \mathbb{M}_2(T))^2.$$

To prove Lemma 7.1, we need the following.

Claim 7.1. *There exist positive constants η_1 , δ and C such that if $\eta_0 \in (0, \eta_1]$ and $\mathbb{M}_1(T) \leq \delta$, then*

$$\|[\partial_y, B_3]\|_{B(Y, Y_1)} \leq C(\mathbb{M}_1(T) + \mathbb{M}_2(T))\langle s \rangle^{-3/4} \text{ for } s \in [0, T].$$

The proof is given in Appendix C.

Claim 7.2. *There exist positive constants η_1 , δ and C such that if $\eta_0 \in (0, \eta_1]$ and $\mathbb{M}_1(T) \leq \delta$, then for $t \in [0, T]$,*

$$\begin{aligned} \|\bar{S}_1 - \tilde{S}_1\|_{B(Y, Y_1)} &\leq C\mathbb{M}_1(T)\langle t \rangle^{-1/4}, \\ \|\bar{S}_3 - \tilde{S}_3\|_{B(Y, Y_1)} &\leq C\mathbb{M}_1(T)\langle t \rangle^{-1/4}e^{-a(4t+L)}. \end{aligned}$$

Claim 7.2 follows immediately from (C.7), (C.8) and Claim 6.1.

Proof of Lemma 7.1. To apply Lemma 4.1, we will transform (6.16) into a system of b and x_y . Let $A(t) = \text{diag}(1, \partial_y)\mathcal{A}(t)\text{diag}(1, \partial_y^{-1})$, $B_5 = B_1 + \partial_y^2 \tilde{S}_1$ and

$$A_0 = \text{diag}(1, \partial_y) \left\{ B_5^{-1}(B_2 - \partial_y^2 \tilde{S}_1)\partial_y^2 + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right\} \text{diag}(1, \partial_y)^{-1},$$

$$A_1(t) = \text{diag}(1, \partial_y)(B_4^{-1} - B_5^{-1})(B_2 - \partial_y^2 \tilde{S}_0)\text{diag}(\partial_y^2, \partial_y) + \text{diag}(1, \partial_y)B_4^{-1}\tilde{\mathcal{A}}_1(t),$$

where $\partial_y^{-1} = \mathcal{F}_\eta^{-1}(i\eta)^{-1}\mathcal{F}_y$. Then $A(t) = A_0 + A_1(t)$. Note that $\tilde{\mathcal{A}}_1(t) = \tilde{\mathcal{A}}_1(t)\text{diag}(1, \partial_y^{-1})$. Multiplying (6.16) by $\text{diag}(1, \partial_y)$ from the left, we can transform (6.16) into

$$(7.2) \quad \partial_t \begin{pmatrix} b \\ x_y \end{pmatrix} = A(t) \begin{pmatrix} b \\ x_y \end{pmatrix} + \sum_{i=1}^8 \text{diag}(1, \partial_y)\mathcal{N}_i.$$

Now we will show that $A(t)$ satisfies the hypothesis (H) of Lemma 4.1. Let $A_0(\eta)$ be the Fourier transform of the operator A_0 . Then

$$\begin{aligned} A_0(\eta) &= \begin{pmatrix} 1 & 0 \\ 0 & i\eta \end{pmatrix} (B_1^{-1} + O(\eta^2))(B_2 + O(\eta^2)) \begin{pmatrix} -\eta^2 & 0 \\ 0 & i\eta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2i\eta & 0 \end{pmatrix} \\ (7.3) \quad &= \begin{pmatrix} -3\eta^2 & 8i\eta \\ i\eta(2 + \mu_3\eta^2) & -\eta^2 \end{pmatrix} + \begin{pmatrix} O(\eta^4) & O(\eta^3) \\ O(\eta^5) & O(\eta^4) \end{pmatrix}, \end{aligned}$$

where $\mu_3 = -\frac{\mu_1}{2} + \frac{3}{4} = \frac{1}{2} + \frac{\pi^2}{24} > 1/8$. Claim D.4 in Appendix D implies $\|A_1(t)\|_{B(Y)} \lesssim e^{-a(4t+L)}$. Thus we prove that $A(t) = A_0 + A_1(t)$ satisfies (H).

Let $U(t, s)$ be the semigroup generated by $A(t)$. By the variation of the constant formula,

$$\begin{pmatrix} b(t) \\ x_y(t) \end{pmatrix} = U(t, 0) \begin{pmatrix} b(0) \\ x_y(0) \end{pmatrix} + \sum_{i=1}^8 \int_0^t U(t, s) \operatorname{diag}(1, \partial_y) \mathcal{N}_i(s) ds.$$

By Lemma 5.3 and Claim D.6,

$$\|b(0)\|_{Y_1} + \|x_y(0)\|_{Y_1} \lesssim \|\tilde{c}(0)\|_{Y_1} + \eta_0 \|\tilde{x}(0)\|_{Y_1} \lesssim \|v_0\|_{X_1}.$$

Applying Lemma 4.1 to the first term of the right hand side, we have for $k \geq 0$,

$$\begin{aligned} \|\partial_y^k b(t)\|_Y + \|\partial_y^{k+1} x(t)\|_Y &\lesssim (1+t)^{-(2k+1)/4} \|v_0\|_{X_1} + \mathfrak{N}_1^k + \mathfrak{N}_2^k, \\ \mathfrak{N}_1^k &= \int_0^t \|\partial_y^k U(t, s) \operatorname{diag}(1, \partial_y) \mathcal{N}_1(s)\|_Y ds, \\ \mathfrak{N}_2^k &= \int_0^t \left\| \partial_y^k U(t, s) \sum_{i=2}^8 \operatorname{diag}(1, \partial_y) \mathcal{N}_i(s) \right\|_Y ds. \end{aligned} \quad (7.4)$$

Now we will estimate \mathfrak{N}_i^k ($i = 1, 2, k = 0, 1, 2$). First, we estimate \mathfrak{N}_1^k . Let $n_1 = 6bx_y$ and $n_2 = 2(\tilde{c} - b) + 3(x_y)^2$. Then $\operatorname{diag}(1, \partial_y) \mathcal{N}_1 = \partial_y \tilde{P}_1^t(n_1, n_2)$. Since $[\partial_y, U(t, s)] = 0$,

$$\partial_y^k U(t, s) \operatorname{diag}(1, \partial_y) \mathcal{N}_1 = \partial_y^{k+1} U(t, s)^t \tilde{P}_1(n_1(s), n_2(s)). \quad (7.5)$$

By (4.2), Claim D.6 and the fact that $[\partial_y, \tilde{P}_1] = 0$,

$$\begin{aligned} \|\tilde{P}_1 n_1\|_{Y_1} + \|\tilde{P}_1 n_2\|_{Y_1} &\lesssim \|b\|_Y \|x_y\|_Y + \|x_y\|_Y^2 + \|b - \tilde{c}\|_{Y_1} \\ &\lesssim (1 + \|\tilde{c}\|_{L^\infty}) \|\tilde{c}\|_Y \|x_y\|_Y + \|\tilde{x}_y\|_Y^2 + \|\tilde{c}\|_Y^2 \\ &\lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-1/2} \quad \text{for } s \in [0, T], \end{aligned} \quad (7.6)$$

$$\begin{aligned} &\|\partial_y \tilde{P}_1 n_1\|_{Y_1} + \|\partial_y \tilde{P}_1 n_2\|_{Y_1} \\ &\lesssim \|b_y\|_Y \|x_y\|_Y + \|b\|_Y \|x_{yy}\|_Y + \|x_y\|_Y \|x_{yy}\|_Y + \|b_y - c_y\|_{Y_1} \\ &\lesssim M_1(T)^2 \langle s \rangle^{-1} \quad \text{for } s \in [0, T], \end{aligned} \quad (7.7)$$

$$\begin{aligned} &\|\partial_y^2 \tilde{P}_1 n_1(s)\|_{Y_1} + \|\partial_y^2 \tilde{P}_1 n_2(s)\|_{Y_1} \\ &\lesssim \|(bx_y)_{yy}\|_{L^1} + \|b_{yy} - \tilde{c}_{yy}\|_{Y_1} + \|((x_y)^2)_{yy}\|_{L^1} \\ &\lesssim \|\tilde{c}\|_Y \|x_{yyy}\|_Y + \|c_y\|_Y \|x_{yy}\|_Y + \|c_{yy}\|_Y \|x_y\|_Y + \|\tilde{c}\|_Y \|c_{yy}\|_Y + \|c_y\|_Y^2 \\ &\quad + \|x_y\|_Y \|x_{yyy}\|_Y + \|x_{yy}\|_Y^2 \\ &\lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for any } s \in [0, T]. \end{aligned} \quad (7.8)$$

Using Lemma 4.1, (7.5) with $k = 0$ and (7.6), we obtain

$$\mathfrak{N}_1^0 \lesssim \mathbb{M}_1(T)^2 \int_0^t \langle t-s \rangle^{-3/4} \langle s \rangle^{-1/2} ds \lesssim \mathbb{M}_1(T)^2 \langle t \rangle^{-1/4} \quad \text{for } t \in [0, T]. \quad (7.9)$$

In the last line, we use Claim 4.1. Using Lemma 4.1, (7.6) for $s \in [0, t/2]$ and (7.7) for $s \in [t/2, t]$, we obtain

$$\begin{aligned}
(7.10) \quad \mathfrak{N}_1^1 &\lesssim \int_0^{t/2} \|\partial_y^2 U(t, s)\|_{B(Y_1, Y)} (\|\tilde{P}_1 n_1(s)\|_{Y_1} + \|\tilde{P}_1 n_2(s)\|_{Y_1}) ds \\
&\quad + \int_{t/2}^t \|\partial_y U(t, s)\|_{B(Y_1, Y)} (\|\partial_y \tilde{P}_1 n_1(s)\|_{Y_1} + \|\partial_y \tilde{P}_1 n_2(s)\|_{Y_1}) ds \\
&\lesssim \mathbb{M}_1(T)^2 \left(\int_0^{t/2} \langle t-s \rangle^{-5/4} \langle s \rangle^{-1/2} ds + \int_{t/2}^t \langle t-s \rangle^{-3/4} \langle s \rangle^{-1} ds \right) \\
&\lesssim \mathbb{M}_1(T)^2 \langle t \rangle^{-3/4} \quad \text{for } t \in [0, T].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(7.11) \quad \mathfrak{N}_1^2 &\lesssim \int_0^{t/2} \|\partial_y^3 U(t, s)\|_{B(Y_1, Y)} (\|\tilde{P}_1 n_1(s)\|_{Y_1} + \|\tilde{P}_1 n_2(s)\|_{Y_1}) ds \\
&\quad + \int_{t/2}^t \|\partial_y U(t, s)\|_{B(Y_1, Y)} (\|\partial_y^2 \tilde{P}_1 n_1(s)\|_{Y_1} + \|\partial_y^2 \tilde{P}_1 n_2(s)\|_{Y_1}) ds \\
&\lesssim \mathbb{M}_1(T)^2 \left(\int_0^{t/2} \langle t-s \rangle^{-7/4} \langle s \rangle^{-1/2} ds + \int_{t/2}^t \langle t-s \rangle^{-3/4} \langle s \rangle^{-5/4} ds \right) \\
&\lesssim \mathbb{M}_1(T)^2 \langle t \rangle^{-1} \quad \text{for } t \in [0, T].
\end{aligned}$$

The rest of nonlinear terms $\sum_{i=2}^8 \text{diag}(1, \partial_y) \mathcal{N}_i$ can be rewritten as a sum of $\mathcal{N}'(t)$ and $\partial_y \mathcal{N}''(t)$ satisfying

$$\begin{aligned}
(7.12) \quad \|\mathcal{N}'(t)\|_{Y_1} &\lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \langle t \rangle^{-5/4} \quad \text{for } t \in [0, T], \\
\|\mathcal{N}''(t)\|_{Y_1} &\lesssim \mathbb{M}_1(T) (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle t \rangle^{-1} \quad \text{for } t \in [0, T].
\end{aligned}$$

First, we prove decay estimates of \mathfrak{N}_2^k ($k = 0, 1, 2$) presuming that (7.12) is true. Then for $t \in [0, T]$ and $0 \leq k \leq 2$,

$$(7.13) \quad \mathfrak{N}_2^k \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \langle t \rangle^{-\min\{1, (2k+1)/4\}}.$$

Indeed, Lemma 4.1 implies that for $t \in [0, T]$,

$$\begin{aligned}
\int_0^t \|\partial_y^k U(t, s) \mathcal{N}'(s)\|_Y ds &\lesssim \int_0^t \|\partial_y^k U(t, s)\|_{B(Y_1, Y)} \|\mathcal{N}'(s)\|_{Y_1} ds \\
&\lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \int_0^t \langle t-s \rangle^{-(2k+1)/4} \langle s \rangle^{-5/4} ds, \\
\int_0^t \|\partial_y^k U(t, s) \partial_y \mathcal{N}''(s)\|_Y ds &\lesssim \int_0^t \|\partial_y^{k+1} U(t, s)\|_{B(Y_1, Y)} \|\mathcal{N}''(s)\|_{Y_1} ds \\
&\lesssim \mathbb{M}_1(T) (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \int_0^t \langle t-s \rangle^{-(2k+3)/4} \langle s \rangle^{-1} ds.
\end{aligned}$$

By Claim 4.1,

$$\int_0^t \langle t-s \rangle^{-(2k+1)/4} \langle s \rangle^{-5/4} ds \lesssim \langle t \rangle^{-(2k+1)/4} \quad \text{for } k = 0, 1, 2,$$

and

$$\int_0^t \langle t-s \rangle^{-(2k+3)/4} \langle s \rangle^{-1} ds \lesssim \begin{cases} \langle t \rangle^{-3/4} \log \langle t \rangle & \text{for } k=0, \\ \langle t \rangle^{-1} & \text{for } k=1, 2. \end{cases}$$

Thus we have (7.13) presuming (7.12).

Now we turn to estimate \mathcal{N}_k ($2 \leq k \leq 8$). First, we will estimate \mathcal{N}_2^k . Let $E_1 = \text{diag}(1, 0)$, $E_2 = \text{diag}(0, 1)$ and

$$\mathcal{N}_{21} = B_3^{-1} E_1 \tilde{P}_1 R^7, \quad \mathcal{N}_{22} = (B_3^{-1} - B_1^{-1}) E_2 \tilde{P}_1 R^7, \quad \mathcal{N}_{23} = B_1^{-1} E_2 \tilde{P}_1 R^7.$$

Then $\mathcal{N}_2 = \mathcal{N}_{21} + \mathcal{N}_{22} + \mathcal{N}_{23}$. Claim 6.2 implies that B_3^{-1} is uniformly bounded on Y and Y_1 for $s \in [0, T]$. Thus by Claim D.7,

$$(7.14) \quad \|\mathcal{N}_{21}\|_{Y_1} \lesssim \|R_1^7\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

Next we will estimate $\|\mathcal{N}_{22}\|_{Y_1}$. Since $(I + \tilde{\mathcal{C}}_2)^{-1} B_1^{-1} E_2 = \frac{1}{2} E_2$, we have $\bar{S}_j B_1^{-1} = \frac{1}{2} \tilde{S}_j$ for $1 \leq j \leq 5$ and $\bar{S}_3 B_1^{-1} E_2 \tilde{P}_1 R^7 = 0$. Thus \mathcal{N}_{22} can be rewritten as

$$\begin{aligned} \mathcal{N}_{22} &= B_3^{-1} (B_1 - B_3) B_1^{-1} E_2 \tilde{P}_1 R^7 \\ &= -B_3^{-1} \left\{ \tilde{\mathcal{C}}_1 - \partial_y^2 (\bar{S}_1 + \bar{S}_2) + \bar{S}_3 + \bar{S}_4 + \bar{S}_5 \right\} B_1^{-1} E_2 \tilde{P}_1 R^7 \\ &= -\frac{1}{2} B_3^{-1} \left\{ \tilde{\mathcal{C}}_1 - \partial_y^2 (\tilde{S}_1 + \tilde{S}_2) + \tilde{S}_4 + \tilde{S}_5 \right\} E_2 R^7. \end{aligned}$$

By Claims B.1–B.5, B.6 and D.7, we have for $s \in [0, T]$ and $j = 1, 2$,

$$\begin{aligned} \|\partial_y^2 \tilde{S}^1 E_2 R^7\|_{Y_1} &\lesssim \eta_0 \|\partial_y S_j^1 [\varphi'_c](R_2^7)\|_{Y_1} \lesssim \|\partial_y R_2^7\|_{Y_1} \lesssim M_1(T)^2 \langle s \rangle^{-5/4}, \\ \|\partial_y^2 \tilde{S}^2 E_2 R^7\|_{Y_1} &\lesssim \eta_0 \|\partial_y S_j^2 [\varphi'_c](R_2^7)\|_{Y_1} \lesssim \|\tilde{c}\|_Y \|\partial_y R_2^7\|_Y + \|c_y\|_Y \|R_2^7\|_Y \\ &\lesssim M_1(T)^2 \langle s \rangle^{-3/2}, \\ \|\tilde{S}_4 E_2 R^7\|_{Y_1} &\lesssim \sum_{j=3,4, k=1,2} \|S_k^j[\psi'](\sqrt{c} - \sqrt{2}) R_2^7\|_{Y_1} \\ (7.15) \quad &\lesssim \left(\|S_k^3[\psi']\|_{B(Y_1)} \|(\sqrt{c} - \sqrt{2}) R_2^7\|_{L^1} + \|S_k^4[\psi']\|_{B(Y, Y_1)} \|(\sqrt{c} - \sqrt{2}) R_2^7\|_{L^2} \right) \\ &\lesssim \mathbb{M}_1(T)^3 e^{-a(4s+L)} \langle s \rangle^{-3/2}, \\ \|\tilde{S}^5 E_2 R^7\|_{Y_1} &\lesssim \sum_{k=1,2} \|S_k^6(R_2^7)\|_{Y_1} \lesssim \|v(t, \cdot)\|_X \|R_2^7\|_Y \lesssim \mathbb{M}_1(T)^2 \mathbb{M}_2(T) \langle s \rangle^{-2}, \\ \|\tilde{\mathcal{C}}_1 E_2 R^7\|_{Y_1} &\leq \|\mathcal{C}_1\|_{B(Y, Y_1)} \|R_2^7\|_Y \lesssim \mathbb{M}_1(T)^3 \langle s \rangle^{-3/2}. \end{aligned}$$

Combining the above, we have

$$(7.16) \quad \|\mathcal{N}_{22}\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

By Claim D.7 and the fact that $\text{diag}(1, \partial_y) \mathcal{N}_{23} = \frac{1}{2} \partial_y(0, R_2^7)$,

$$(7.17) \quad \|\text{diag}(1, \partial_y) \mathcal{N}_{23}\|_{Y_1} \lesssim \|\partial_y R_2^7\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

Combining (7.14), (7.16) and (7.17), we have

$$(7.18) \quad \|\text{diag}(1, \partial_y) \mathcal{N}_2\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

Next, we will estimate \mathcal{N}_3 . Claims D.2 and D.5 imply that for $s \in [0, T]$,

$$\|R^3\|_{Y_1} + \|R^4\|_{Y_1} + \|R^6\|_{Y_1} \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \langle s \rangle^{-3/2}.$$

Using Claims B.3, B.4 and (4.2), we can show that for $s \in [0, T]$,

$$\left\| \tilde{S}_4 \begin{pmatrix} 0 \\ \tilde{c} \end{pmatrix} \right\|_{Y_1} \lesssim \sum_{\substack{j=3,4 \\ k=1,2}} \|S_k^j[\psi'](\sqrt{c} - \sqrt{2})\tilde{c}\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 e^{-a(4t+L)} \langle s \rangle^{-1/2}$$

in the same way as (7.15). Thus for $s \in [0, T]$,

$$(7.19) \quad \|\mathcal{N}_3(s)\|_{Y_1} \lesssim \|B_3^{-1}\|_{B(Y_1)} \|\tilde{R}^1\|_{Y_1} \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \langle s \rangle^{-3/2}.$$

Next we will estimate \mathcal{N}_4^k . Let $\mathcal{N}_{41} = B_3^{-1} \tilde{R}^2$ and $\mathcal{N}_{42} = [B_3^{-1}, \partial_y] \tilde{R}^2$. Then $\mathcal{N}_4 = \partial_y \mathcal{N}_{41} + \mathcal{N}_{42}$. By Claims D.1, D.5 and (C.6),

$$(7.20) \quad \|\tilde{R}^2\|_{Y_1} \lesssim \mathbb{M}_1(T)(\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-1} \quad \text{for } s \in [0, T].$$

By Claims 6.2 and 7.1,

$$(7.21) \quad \|[B_3^{-1}, \partial_y]\|_{B(Y, Y_1)} \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-3/4} \quad \text{for } s \in [0, T].$$

Combining (7.20), (7.21) and Claim 6.2, we have for $s \in [0, T]$,

$$(7.22) \quad \|\mathcal{N}_{41}(s)\|_{Y_1} \lesssim \mathbb{M}_1(T)(\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-1},$$

$$(7.23) \quad \|\mathcal{N}_{42}(s)\|_{Y_1} \lesssim \mathbb{M}_1(T)(\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \langle s \rangle^{-7/4}.$$

Next we will estimate \mathcal{N}_5 . Let $r_8 = \tilde{P}_1\{(I + \mathcal{C}_2)(c_y x_y) - (bx_y)_y\}$. Then

$$\|r_8\|_Y \lesssim \|r_8\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-1}, \quad \|\partial_y r_8\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4}.$$

Here we use Claims 6.1 and B.7. By (6.15) and (C.6),

$$\begin{aligned} \|R^8\|_{Y_1} &\lesssim \|\tilde{S}_1\|_{B(Y_1)} \|\partial_y r_8\|_{Y_1} + \left(\|\partial_y, \tilde{S}_1\|_{B(Y, Y_1)} + \sum_{j=2,4,5} \|\tilde{S}_j\|_{B(Y, Y_1)} \right) \|r_8\|_Y \\ &\quad + \|\tilde{S}_3\|_{B(Y_1)} \|r_8\|_{Y_1}. \end{aligned}$$

Combining the above with (C.1)–(C.5) and (C.10) in Appendix C, we have

$$\|R^8\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 (1 + \mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

Since $\|\partial_y^2 \tilde{S}_0\|_{B(Y_1)} \lesssim \eta_0^2$ by Claim B.1 and (C.6), it follows from Claim D.6 that

$$\|R^9\|_{Y_1} \lesssim \|b_{yy} - c_{yy}\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

By Claims D.3 and D.6,

$$\|R^{10}\|_{Y_1} \lesssim e^{-a(4s+L)} \|b - \tilde{c}\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-1/2} e^{-a(4s+L)} \quad \text{for } s \in [0, T].$$

Thus we have

$$(7.24) \quad \|\mathcal{N}_5\|_{Y_1} \lesssim \|\tilde{R}^3(s)\| \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

Next we will estimate \mathcal{N}_6^k . Since the second column of $\tilde{A}_1(t)$ is 0,

$$\mathcal{N}_6 = (B_3^{-1} - B_4^{-1}) \tilde{A}_1(t) \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

By the definitions of B_3 and B_4 ,

$$B_3 - B_4 = \tilde{\mathcal{C}}_1 + \partial_y^2 (\tilde{S}_1 - \tilde{S}_1) + \partial_y^2 \tilde{S}_2 - (\tilde{S}_3 - \tilde{S}_3) - \tilde{S}_4 - \tilde{S}_5.$$

Hence it follows from Claims 6.1, 7.2, (C.2) and (C.4)–(C.6) that

$$\begin{aligned} \|B_4 - B_3\|_{B(Y, Y_1)} &\leq \|\tilde{\mathcal{C}}_1\|_{B(Y, Y_1)} + \sum_{j=1,3} \|\partial_y^2(\bar{S}_j - \tilde{S}_j)\|_{B(Y, Y_1)} + \sum_{j=2,4,5} \|\partial_y^2 \bar{S}_j\|_{B(Y, Y_1)} \\ &\lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-1/4} \quad \text{for } s \in [0, T]. \end{aligned}$$

In view of Claims 6.2 and 6.3 and the above, we have for $s \in [0, T]$,

$$\begin{aligned} (7.25) \quad \|B_3^{-1} - B_4^{-1}\|_{B(Y, Y_1)} &\leq \|B_4^{-1}\|_{B(Y_1)} \|B_4 - B_3\|_{B(Y, Y_1)} \|B_3^{-1}\|_{B(Y)} \\ &\lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-1/4}. \end{aligned}$$

Combining Claim D.3 and (7.25), we have for $s \in [0, T]$,

$$\begin{aligned} (7.26) \quad \|\mathcal{N}_6\|_{Y_1} &\lesssim \|B_3^{-1} - B_4^{-1}\|_{B(Y, Y_1)} \|\tilde{\mathcal{A}}_1(t)\|_{B(Y)} \|b\|_Y \\ &\lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \mathbb{M}_1(T) \langle s \rangle^{-1/2} e^{-a(4s+L)}. \end{aligned}$$

Since $\|\tilde{S}_0\|_{B(Y)} \lesssim 1$ by Claim B.1, it follows from (C.6) and (7.25) that for $s \in [0, T]$,

$$(7.27) \quad \|\mathcal{N}_7\|_{Y_1} \lesssim \|B_3^{-1} - B_4^{-1}\|_{B(Y, Y_1)} \|b_{yy}\|_Y \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \mathbb{M}_1(T) \langle s \rangle^{-5/4}.$$

Finally, we will estimate \mathcal{N}_8^k . Let

$$\begin{aligned} \mathcal{N}_{81} &= (B_4^{-1} - B_3^{-1}) \partial_y^2 \tilde{S}_0 \begin{pmatrix} 0 \\ x_{yy} \end{pmatrix}, \quad \tilde{R}^4 = B_4^{-1} B_2 \begin{pmatrix} 0 \\ x_{yy} \end{pmatrix}, \\ \mathcal{N}_{82} &= B_3^{-1} (B_4 - B_3 + \tilde{\mathcal{C}}_1) \tilde{R}^4, \quad \mathcal{N}_{83} = B_1^{-1} (B_3 - B_1 - \tilde{\mathcal{C}}_1) B_3^{-1} \tilde{\mathcal{C}}_1 \tilde{R}^4, \\ \mathcal{N}_{84} &= -B_1^{-1} \tilde{\mathcal{C}}_1 (I - B_3^{-1} \tilde{\mathcal{C}}_1) \tilde{R}^4. \end{aligned}$$

Then $\mathcal{N}_8 = \sum_{1 \leq j \leq 4} \mathcal{N}_{8j}$. Since $[\partial_y, \tilde{S}_0] = 0$, we have

$$\left\| \partial_y \tilde{S}_0 \begin{pmatrix} 0 \\ x_{yy} \end{pmatrix} \right\|_Y \lesssim \|x_{yyy}\|_Y.$$

Combining the above with (C.6) and (7.25), we see that for $s \in [0, T]$,

$$\begin{aligned} (7.28) \quad \|\mathcal{N}_{81}\|_{Y_1} &\lesssim \eta_0 (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-1/4} \|x_{yyy}\|_Y \\ &\lesssim \eta_0 (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \mathbb{M}_1(T) \langle s \rangle^{-5/4}. \end{aligned}$$

Next we will estimate \mathcal{N}_{82} . Let

$$n_3 = \partial_y (\bar{S}_1 - \tilde{S}_1 + \bar{S}_2) \partial_y \tilde{R}^4, \quad n_4 = \partial_y [\partial_y, \bar{S}_1 + \bar{S}_2] \tilde{R}^4, \quad n_5 = (\bar{S}_3 - \tilde{S}_3 + \bar{S}_4 + \bar{S}_5) \tilde{R}^4.$$

Then $\mathcal{N}_{82} = B_3^{-1} (n_5 - n_3 - n_4)$. Claim 6.3 and the fact that $[\partial_y, B_4] = 0$ imply that for $s \in [0, T]$,

$$\begin{aligned} (7.29) \quad \|\tilde{R}^4\|_Y &\lesssim \|x_{yy}\|_Y \lesssim \mathbb{M}_1(T) \langle s \rangle^{-3/4}, \\ \|\partial_y \tilde{R}^4\|_Y &\lesssim \|x_{yyy}\|_Y \lesssim \mathbb{M}_1(T) \langle s \rangle^{-1}. \end{aligned}$$

We see that $\|n_3\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4}$ follows from Claim 7.2, (C.2), (C.6) and (7.29) and that $\|n_4\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-3/2}$ follows from (C.10), (C.11), (C.6) and (7.29). By Claim 7.2, (C.4), (C.5) and (7.29),

$$\|n_5\|_{Y_1} \lesssim \mathbb{M}_1(T) \langle s \rangle^{-3/4} (\mathbb{M}_1(T) e^{-a(4s+L)} \langle s \rangle^{-1/4} + \mathbb{M}_2(T) \langle s \rangle^{-3/4}).$$

Thus we have

$$(7.30) \quad \|\mathcal{N}_{82}\|_{Y_1} \lesssim \mathbb{M}_1(T) (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

Next we will estimate \mathcal{N}_{83} . Let

$$\begin{aligned} n_6 &= [\partial_y, \bar{S}_1 + \bar{S}_2] B_3^{-1} \tilde{\mathcal{C}}_1 \tilde{R}^4, & n_7 &= (\bar{S}_1 + \bar{S}_2) [\partial_y, B_3^{-1}] \tilde{\mathcal{C}}_1 \tilde{R}^4, \\ n_8 &= (\bar{S}_1 + \bar{S}_2) B_3^{-1} \partial_y \tilde{\mathcal{C}}_1 \tilde{R}^4, & n_9 &= (\bar{S}_3 + \bar{S}_4 + \bar{S}_5) B_3^{-1} \tilde{\mathcal{C}}_1 \tilde{R}^4. \end{aligned}$$

Then $\mathcal{N}_{83} = B_1^{-1} \partial_y (n_6 + n_7 + n_8) - B_1^{-1} n_9$. By Claims 6.1, 6.2, (C.10) and (C.11),

$$\begin{aligned} \|n_6\|_{Y_1} &\lesssim \sum_{j=1,2} \|[\partial_y, \bar{S}_j]\|_{B(Y, Y_1)} \|B_3^{-1}\|_{B(Y)} \|\tilde{\mathcal{C}}_1\|_{B(Y)} \|\tilde{R}^4\|_Y Y_1 \\ &\lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-2} \quad \text{for } s \in [0, T]. \end{aligned}$$

Using Claims 6.2, 6.2, (C.1)–(C.5), (7.29) and (C.9), we can obtain

$$\begin{aligned} \|n_7\|_{Y_1} &\lesssim \mathbb{M}_1(T)^2 (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-2}, & \|n_8\|_{Y_1} &\lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4}, \\ \|n_9\|_{Y_1} &\lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-1} (e^{-a(4s+L)} + \mathbb{M}_2(T) \langle s \rangle^{-3/4}). \end{aligned}$$

Combining the above, we have

$$(7.31) \quad \|\mathcal{N}_{83}\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4}.$$

Since $\text{diag}(1, \partial_y) B_1^{-1} \tilde{\mathcal{C}}_1 = \frac{1}{2} \partial_y \tilde{\mathcal{C}}_1$,

$$2 \text{diag}(1, \partial_y) \mathcal{N}_{84} = \left\{ [\partial_y, \tilde{\mathcal{C}}_1] B_3^{-1} + \tilde{\mathcal{C}}_1 [\partial_y, B_3^{-1}] + (\tilde{\mathcal{C}}_1 B_3^{-1} - I) \partial_y \right\} \tilde{\mathcal{C}}_1 \tilde{R}^4.$$

Combining Claims 6.2, B.6, B.7 and 7.1 with (7.29), we obtain

$$(7.32) \quad \|\text{diag}(1, \partial_y) \mathcal{N}_{84}\|_{Y_1} \lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4} \quad \text{for } s \in [0, T].$$

Thus for $s \in [0, T]$,

$$(7.33) \quad \sum_{1 \leq j \leq 4} \|\text{diag}(1, \partial_y) \mathcal{N}_{8j}\| \lesssim \mathbb{M}_1(T) (\mathbb{M}_1(T) + \mathbb{M}_2(T)) \langle s \rangle^{-5/4}$$

follows from (7.28), (7.30), (7.31) and (7.32).

Now let $\mathcal{N}' = \text{diag}(1, \partial_y) \left(\sum_{\substack{2 \leq i \leq 8 \\ i \neq 4}} \mathcal{N}_i + \mathcal{N}_{42} \right)$ and $\mathcal{N}'' = \text{diag}(1, \partial_y) \mathcal{N}_{41}$. Then (7.12) follows immediately from (7.18), (7.19), (7.22), (7.23), (7.24), (7.26), (7.27) and (7.33). By (7.4) and Claim D.6,

$$(7.34) \quad \begin{aligned} \|\partial_y^k \tilde{c}(t)\|_Y + \|\partial_y^{k+1} x(t)\|_Y &\lesssim \langle t \rangle^{-(2k+1)/4} \|v_0\|_{X_1} + \mathfrak{N}_1^k + \mathfrak{N}_2^k \\ &\quad + \mathbb{M}_1(T)^2 \langle t \rangle^{-\min\{(2k+3)/4, 3/2\}}. \end{aligned}$$

for $0 \leq k \leq 2$ and $t \in [0, T]$. Combining (7.9)–(7.11) and (7.13) with (7.34), we obtain (7.1). This completes the proof of Lemma 7.1. \square

8. L^2 BOUND ON $v(t, z, y)$

In this section, we will estimate $\mathbb{M}_3(T)$ assuming that $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$ are small. First, we will show a variant of the L^2 conservation law on v .

Lemma 8.1. *Let $a \in (0, 2)$ and $T > 0$. Suppose $v(t) \in C([0, T]; X \cap L^2(\mathbb{R}^2))$ is a solution of (6.1) and that $v(t)$, $c(t)$ and $x(t)$ satisfy (5.3), (5.9) and (5.10). Then*

$$Q(t, v) := \int_{\mathbb{R}^2} \{v(t, z, y)^2 - 2\psi_{c(t, y), L}(z + 4t)v(t, z, y)\} \, dz dy$$

satisfies for $t \in [0, T]$,

$$\begin{aligned} Q(t, v) = & Q(0, v) + 2 \int_0^t \int_{\mathbb{R}^2} \left(\ell_{11} + \ell_{12} + 6\varphi'_{c(s,y)}(z) \tilde{\psi}_{c(s,y)}(z) \right) v(s, z, y) dz dy ds \\ & - 2 \int_0^t \int_{\mathbb{R}^2} \ell \psi_{c(t,y),L}(z + 4t) dz dy ds - 6 \int_{\mathbb{R}^2} \varphi'_{c(t,y)}(z) v(t, z, y)^2 dz dy \\ & - 6 \int_{\mathbb{R}^2} v(t, z, y) \left\{ c_{yy}(t, y) \int_{-\infty}^z \partial_c \varphi_{c(t,y)}(z_1) dz_1 \right. \\ & \quad \left. + c_y(t, y)^2 \int_{-\infty}^z \partial_c^2 \varphi_{c(t,y)}(z_1) dz_1 \right\} dz dy. \end{aligned}$$

Proof. Suppose $v_0 \in H^3(\mathbb{R}^2)$ and $\partial_x^{-1} v_0 \in H^2(\mathbb{R}^2)$. Then as in Section 6, we have $v(t) \in C([0, T]; H^3(\mathbb{R}^2))$ and $\partial_x^{-1} \partial_y v(t) \in C([0, T]; H^1(\mathbb{R}^2))$. Using (6.1), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} v(t, x, y)^2 dx dy = & 2 \int v \left(\mathcal{L}_{c(t,y)} v + \ell + \partial_z(N_1 + N_2) + N_3 \right) dz dy \\ = & 2 \int \ell v dz dy + 6 \int (\tilde{\psi}'_c - \varphi'_c) v^2 dz dy, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \psi_{c(t,y),L}(z + 4t) v(t, z, y) dz dy = & \int \left(c_t \partial_c \tilde{\psi}_c + 4 \tilde{\psi}'_c \right) v \\ & + \int \tilde{\psi}_c \left\{ \mathcal{L}_{c(t,y)} v + \ell + \partial_z(N_1 + N_2) + N_3 \right\}. \end{aligned}$$

Since $(\partial_z^{-1} \partial_y^2) v(z, y) = - \int_z^\infty \partial_y^2 v(z_1, y) dz_1$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{\psi}_c \mathcal{L}_c v dz dy = & - \int \tilde{\psi}_c \partial_z (\partial_z^2 - 2c + 6\varphi_c) v dz dy - 3 \int \tilde{\psi}_c \partial_z^{-1} \partial_y^2 v dz dy \\ = & \int v (\tilde{\psi}_c''' - 2c \tilde{\psi}_c' + 6\varphi_c \tilde{\psi}_c') dz dy \\ & + 3 \int_{\mathbb{R}^2} v \left\{ c_{yy} \int_{-\infty}^z \partial_c \tilde{\psi}_c + (c_y)^2 \int_{-\infty}^z \partial_c^2 \tilde{\psi}_c \right\} dz dy, \end{aligned}$$

where $\partial_c^k \tilde{\psi}_c = \partial_c^k \psi_{c(t,y),L}(x + 4t)$ for $k \geq 0$. By integration by parts, we have

$$\int_{\mathbb{R}^2} \tilde{\psi}_c \partial_z N_1 dz dy = 3 \int \tilde{\psi}'_c v^2 dz dy,$$

and

$$\begin{aligned} & \int (\partial_z N_2 + N_3) \tilde{\psi}_c dz dy \\ = & - \int v \left\{ (x_t - 2c - 3(x_y)^2) \tilde{\psi}'_c + 3x_{yy} \tilde{\psi}_c + 6c_y x_y \partial_c \tilde{\psi}_c + 6 \tilde{\psi}'_c \tilde{\psi}_c \right\} dz dy. \end{aligned}$$

Combining the above, we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^2} \{v(t, z, y)^2 - 2\psi_{c(t,y),L}(z+4t)v(t, z, y)\} dz dy \\
&= 2 \int_{\mathbb{R}^2} \left\{ \left(\ell_{11} + \ell_{12} + 6\varphi'_{c(t,y)}(z)\tilde{\psi}_{c(t,y)}(z) \right) v(t, z, y) - \ell\psi_{c(t,y),L}(z+4t) \right\} dz dy \\
&+ 6 \int_{\mathbb{R}^2} v(t, z, y) \left\{ c_{yy}(t, y) \int_{\mathbb{R}} (\partial_c \varphi_{c(t,y)}(z_1) - \partial_c \psi_{c(t,y),L}(z_1+4t)) dz_1 \right\} dz dy \\
&+ 6 \int_{\mathbb{R}^2} v(t, z, y) \left\{ c_y(t, y)^2 \int_{\mathbb{R}} (\partial_c^2 \varphi_{c(t,y)}(z_1) - \partial_c^2 \psi_{c(t,y),L}(z_1+4t)) dz_1 \right\} dz dy \\
&- 6 \int_{\mathbb{R}^2} \ell_{13}^* v(t, z, y) dz dy - 6 \int_{\mathbb{R}^2} \varphi'_{c(t,y)}(z) v(t, z, y)^2 dz dy,
\end{aligned}$$

where

$$\ell_{13}^* = c_{yy}(t, y) \int_{-\infty}^z \partial_c \varphi_{c(t,y)}(z_1) dz_1 + c_y(t, y)^2 \int_{-\infty}^z \partial_c^2 \varphi_{c(t,y)}(z_1) dz_1.$$

Since $\int_{\mathbb{R}} \partial_c^k \varphi_c(x) dx = \int_{\mathbb{R}} \partial_c^k \psi_{c,L}(x) dx$ for $k \geq 0$ by (5.2), we see that Lemma 8.1 holds provided v_0 and $\partial_x^{-1} \partial_y v_0$ are smooth.

For general $v_0 \in X \cap L^2(\mathbb{R}^2)$, we can prove Lemma 8.1 by a standard limiting argument. The mapping

$$(8.1) \quad L^2(\mathbb{R}^2) \ni v_0 \mapsto \tilde{v}(t) \in C([0, T]; L^2(\mathbb{R}^2))$$

is continuous for any $T > 0$ by [29]. On the other hand, it follows from (E.4) that a solution $\tilde{v}(t)$ of (E.1) satisfies $\sup_{t \in [0, T]} \|\tilde{v}(t)\|_X \leq C$, where C is a constant depending only on T , $\|\tilde{v}(0)\|_{L^2(\mathbb{R}^2)}$ and $\|\tilde{v}(0)\|_X$. Thus the mapping

$$(8.2) \quad X \cap L^2(\mathbb{R}^2) \ni v_0 \mapsto \tilde{v}(t) \in C([0, T]; L^2(\mathbb{R}^2; e^{ax} dx dy))$$

is continuous since $\|u\|_{L^2(\mathbb{R}^2; e^{ax} dx dy)} \lesssim \|u\|_{L^2(\mathbb{R}^2)}^{1/2} \|u\|_X^{1/2}$ for every $u \in X \cap L^2(\mathbb{R}^2)$. If η_0 is sufficiently small, it is clear from Lemma 5.2 and Remark 5.1 that $(\tilde{c}(t), \tilde{x}(t)) \in Y \times Y$ as well as its time derivate depends continuously on $v(t) \in L^2(\mathbb{R}^2; e^{ax} dx dy)$. This completes the proof of Lemma 8.1. \square

Using Lemma 8.1, we will estimate the upper bound of $\|v(t)\|_{L^2}$.

Lemma 8.2. *Let $a \in (0, 1)$ and δ_4 be as in Lemma 7.1. Then there exists a positive constant C such that if $\mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0 + e^{-aL} \leq \delta_4$, then*

$$\mathbb{M}_3(T) \leq C(\|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_1(T) + \mathbb{M}_2(T)).$$

Proof. Remark 5.1 and Proposition 5.4 tell us that we can apply Lemma 8.1 for $t \in [0, T]$ if $\mathbb{M}_1(T)$ and $\mathbb{M}_2(T)$ are sufficiently small.

Since we have for $j, k \geq 0$ and $z \in \mathbb{R}$,

$$(8.3) \quad \partial_z^j \partial_c^k \varphi_c(z) \lesssim e^{-2a|z|}, \quad \int_{-\infty}^z \partial_c^j \varphi_c(z_1) dz_1 \lesssim \min(1, e^{2az}),$$

it follows that

$$\begin{aligned}
(8.4) \quad & \left| \int_{\mathbb{R}^2} (\ell_{11} + \ell_{12} - \ell_{13}^*) v dz dy \right| \\
& \lesssim (\|c_t - 6c_{xy} x_y\|_{L^2} + \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|x_{yy}\|_{L^2} + \|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2) \|v\|_X,
\end{aligned}$$

$$(8.5) \quad \left| \int_{\mathbb{R}^2} \varphi'_{c(t,y)}(z) v(t, z, y)^2 dz dy \right| \lesssim \|v\|_X^2,$$

$$(8.6) \quad \left| \int_{\mathbb{R}^2} \varphi'_{c(t,y)}(z) \tilde{\psi}_{c(t,y)}(z) v(t, z, y) dz dy \right| \lesssim \|e^{az} \tilde{\psi}_{c(t,y)}\|_{L_{yz}^2} \|v\|_X e^{-a(4t+L)}.$$

Next, we will estimate $\int_{\mathbb{R}^2} \ell \tilde{\psi}_{c(t,y)}$. In view of (8.3), we see that $\ell_{11} + \ell_{12}$ is exponentially localized and that

$$(8.7) \quad \begin{aligned} & \left| \int_{\mathbb{R}^2} (\ell_{11} + \ell_{12}) \tilde{\psi}_{c(t,y)}(t, z, y) dz dy \right| \\ & \leq \|e^{-az} (\ell_{11} + \ell_{12})\|_{L_{yz}^2} \|e^{az} \tilde{\psi}_{c(t,y)}\|_{L_{yz}^2} \\ & \lesssim (\|c_t - 6c_y x_y\|_{L^2} + \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|x_{yy}\|_{L^2}) \|e^{az} \tilde{\psi}_{c(t,y)}\|_{L_{yz}^2}. \end{aligned}$$

By integration by parts and the fact that $\|\partial_c \tilde{\psi}_{c(t,y)}(z)\|_{L_y^\infty L_z^2} \lesssim 1$,

$$(8.8) \quad \int_{\mathbb{R}^2} \ell_{21} \tilde{\psi}_{c(t,y)}(t, z, y) dz dy = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \psi_{c(t,y)}(z)^2 dz dy + O(\|c_y x_y\|_{L^2} \|\tilde{c}\|_{L^2}).$$

Similarly,

$$(8.9) \quad \begin{aligned} & \left| \int_{\mathbb{R}^2} \ell_{22} \tilde{\psi}_{c(t,y)} dz dy \right| = 3 \left| \int_{\mathbb{R}^2} \left\{ \varphi'_{c(t,y)}(z) - x_{yy}(t, y) \right\} \tilde{\psi}_{c(t,y)}(z)^2 dz dy \right| \\ & \lesssim \|e^{az} \tilde{\psi}\|_{L_{yz}^2}^2 + \|x_{yy}\|_{L^\infty} \|\psi_{c(t,y),L}\|_{L_z^2 L_y^2}^2. \end{aligned}$$

Since $\|\ell_{13}\|_{L_z^\infty L_y^2} + \|\ell_{23}\|_{L_z^\infty L_y^2} \lesssim \|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2$ and $\|\psi_{c(t,y),L}\|_{L_z^1 L_y^2} = O(\|\tilde{c}\|_{L^2})$,

$$(8.10) \quad \sum_{j=1,2} \left| \int_{\mathbb{R}^2} \ell_{j3} \tilde{\psi}_{c(t,y)} dz dy \right| \lesssim \|\tilde{c}\|_{L^2} (\|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2).$$

In view of the definition of $\tilde{\psi}$,

$$(8.11) \quad \begin{aligned} & \|\tilde{\psi}_{c(t,y)}\|_X \lesssim \|\tilde{c}\|_{L^2} e^{-a(4t+L)}, \\ & \|\tilde{\psi}_{c(t,y)}\|_{L^2(\mathbb{R}^2)} = 2\sqrt{2} \|\sqrt{c} - \sqrt{2}\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \lesssim \|\tilde{c}\|_{L^2}. \end{aligned}$$

Claims D.3, D.6 and (6.16) imply that for $t \in [0, T]$,

$$\begin{aligned} & \|c_t\|_Y + \|x_t - 2c - 3(x_y)^2\|_{L^2} \lesssim \|b_t\|_Y + \|x_t - 2c - 3(x_y)^2\|_{L^2} \\ & \lesssim \|c_{yy}\|_Y + \|x_{yy}\|_Y + \|\tilde{\mathcal{A}}_1(t)\|_{B(Y)} \|b\|_Y + \|(bx_y)_y\|_Y + \sum_{i=2}^8 \|\mathcal{N}_i\|_Y \\ & \lesssim \mathbb{M}_1(T) \langle t \rangle^{-3/4} + \sum_{i=2}^8 \|\mathcal{N}_i\|_Y. \end{aligned}$$

Following the proof of Lemma 7.1, we see that

$$\sum_{2 \leq i \leq 8} \|\mathcal{N}_i\|_Y \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \langle t \rangle^{-1}.$$

Thus we have

$$(8.12) \quad \|c_t\|_Y + \|x_t - 2c - 3(x_y)^2\|_{L^2} \lesssim \mathbb{M}_1(T) \langle t \rangle^{-3/4} + (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \langle t \rangle^{-1}.$$

Combining (8.4)–(8.10), (8.11) and (8.12) with Lemma 8.1, we see that for $t \in (0, T]$,

$$(8.13) \quad \begin{aligned} & \left[Q(s, v) + 8\|\psi\|_{L^2}^2 \|\sqrt{c(s)} - \sqrt{2}\|_{L^2(\mathbb{R})}^2 \right]_{s=0}^{s=t} \\ & \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2 \int_0^t \langle s \rangle^{-5/4} ds \lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))^2. \end{aligned}$$

Since $\|\sqrt{c(0)} - \sqrt{2}\|_{L^2} \lesssim \|\tilde{c}(0)\|_Y \lesssim \|v_0\|_{X_1}$ and

$$Q(t, v) = \|v(t)\|_{L^2(\mathbb{R}^2)}^2 + O(\|\tilde{c}(t)\|_Y \|v(t)\|_{L^2(\mathbb{R}^2)}),$$

Lemma 8.2 follows immediately from (8.13). Thus we complete the proof. \square

9. LOW FREQUENCIES BOUND OF $v(t, x, y)$ IN y

Let $v_1(t) = P_1(0, 2M)v(t)$. Since $v_1(t)$ does not include high frequency modes in the y variable, we can estimate $v_1(t)$ in the similar mannar as generalized KdV equations ([32]) by using the semigroup estimates obtained in Section 3. In this section, we will estimate $v_1(t)$ in the exponentially weighted space X .

Lemma 9.1. *Let η_0 , a and M be positive constants satisfying $\nu_0 < a < 2$ and $\nu(2M) > a$. Suppose that $v(t)$ is a solution of (6.1). Then there exist positive constants b_1 , δ_5 and C such that if $\mathbb{M}_1(T) + \mathbb{M}_2(T) < \delta_5$, then for $t \in [0, T]$,*

$$\|v_1(t, \cdot)\|_X \leq C e^{-2b_1\eta_0^2 t} \|v(0, \cdot)\|_X + \left\{ \mathbb{M}_1(T) + \mathbb{M}_2(T) \sum_{i=1}^3 \mathbb{M}_i(T) \right\} \langle t \rangle^{-3/4}.$$

Let $\chi(\eta)$ be a nonnegative smooth function such that $\chi(\eta) = 1$ if $|\eta| \leq 1$ and $\chi(\eta) = 0$ if $|\eta| \geq 2$. Let $\chi_M(\eta) = \chi(\eta/M)$ and

$$P_{\leq M} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_M(\eta) \hat{u}(\xi, \eta) e^{i(x\xi + y\eta)} d\xi d\eta, \quad P_{\geq M} = I - P_{\leq M}.$$

To estimate $v_1(t)$, we need the following.

Claim 9.1. *There exists a positive constant C such that*

$$(9.1) \quad \|P_{\leq M} u\|_{L_x^1 L_y^2} \leq C \sqrt{M} \|u\|_{L^1(\mathbb{R}^2)}.$$

Proof. Applying Young's inequality to

$$(9.2) \quad (P_{\leq M} u)(x, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}^{-1}(\chi_M)(y - y_1) u(x, y_1) dy_1,$$

we have

$$\|P_{\leq M} u\|_{L_y^2} \leq \|\mathcal{F}^{-1}(\chi_M)\|_{L^2(\mathbb{R})} \|u(x, \cdot)\|_{L^1(\mathbb{R})} \lesssim \sqrt{M} \|u(x, \cdot)\|_{L^1(\mathbb{R})}.$$

Integrating the above over \mathbb{R} in x , we obtain (9.1). \square

Proof of Lemma 9.1. Let $v_2(t) = P_2(\eta_0, M)v(t)$. Then

$$(9.3) \quad \partial_t v_2 = \mathcal{L} v_2 + P_2(\eta_0, 2M) \{ \ell + \partial_x (N_1 + N_2 + N'_2) + N_3 \},$$

where $N_2' = 2\tilde{c}(t, y)v(t, z, y) + 6(\varphi(z) - \varphi_{c(t, y)}(z))v(t, z, y)$. Hereafter we abbreviate $P_2(\eta_0, 2M)$ as P_2 . By Proposition 3.2 and Corollary 3.3, we may assume that

$$\begin{aligned} \|e^{t\mathcal{L}}P_2f\|_X &\leq Ke^{-2b_1\eta_0^2t}\|f\|_X, \\ \|e^{t\mathcal{L}}P_2\partial_zf\|_X &\leq K(1+t^{-1/2})e^{-2b_1\eta_0^2t}\|f\|_X, \\ \|e^{t\mathcal{L}}P_2\partial_zf\|_X &\leq K(1+t^{-3/4})e^{-2b_1\eta_0^2t}\|e^{az}f\|_{L_z^1L_y^2}, \end{aligned}$$

where K and b_1 are positive constants independent of $f \in X$ and $t > 0$. Applying the semigroup estimates Lemma 3.2 and Corollary 3.3 to (9.3), we have

$$\begin{aligned} \|v_2(t)\|_X &\lesssim e^{-2b_1\eta_0^2t}\|v_2(0)\|_X + \int_0^t e^{-2b_1\eta^2(t-s)}(t-s)^{-3/4}\|e^{az}N_1(s)\|_{L_z^1L_y^2}ds \\ &\quad + \int_0^t e^{-2b_1\eta^2(t-s)}(t-s)^{-1/2}(\|N_2(s)\|_X + \|N_2'(s)\|_X)ds \\ &\quad + \int_0^t e^{-2b_0\eta^2(t-s)}(\|\ell(s)\|_X + \|N_3(s)\|_X)ds. \end{aligned}$$

By Claim 9.1,

$$\begin{aligned} \|e^{az}P_2N_1\|_{L_z^1L_y^2} &\lesssim \sqrt{M}\|v\|_{L^2}\|v\|_X \\ &\lesssim \sqrt{M}\mathbb{M}_2(T)\mathbb{M}_3(T)\langle t \rangle^{-3/4} \quad \text{for } t \in [0, T]. \end{aligned}$$

By (8.12), we have for $t \in [0, T]$,

$$\begin{aligned} (9.4) \quad \|\ell_1\|_X &\lesssim \|x_t - 2c - 3(x_y)^2\|_{L^2} + \|c_t - 6c_yx_y\|_{L^2} + \|x_{yy}\|_{L^2} + \|c_{yy}\|_{L^2} + \|c_y\|_{L^4}^2 \\ &\lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T)^2)\langle t \rangle^{-3/4}, \end{aligned}$$

$$\begin{aligned} (9.5) \quad \|\ell_2\|_X &\lesssim e^{-a(4t+L)}(\|c_t - 6c_yx_y\|_Y + \|x_t - 2c - 3(x_y)^2\|_Y + \|\tilde{c}\|_Y + \\ &\quad + \|x_{yy}\|_Y + \|c_{yy}\|_Y + \|c_y\|_{L^4}^2) \\ &\lesssim e^{-a(4t+L)}(\mathbb{M}_1(T) + \mathbb{M}_2(T)^2)\langle t \rangle^{-1/4}, \end{aligned}$$

and

$$\begin{aligned} \|N_2\|_X + \|N_2'\|_X &\lesssim (\|x_t - 2c - 3(x_y)^2\|_{L^\infty} + \|\tilde{c}\|_{L^\infty})\|v\|_X \\ &\lesssim (\mathbb{M}_1(T) + \mathbb{M}_2(T))\mathbb{M}_2(T)\langle t \rangle^{-5/4}. \end{aligned}$$

Here we use $\sup_{y,z}(|\varphi_{c(t,y)}(z) - \varphi(z)| + |\tilde{\psi}_{c(t,y)}|) \lesssim \|\tilde{c}(t)\|_{L^\infty}$. Since $\|\partial_y P_2\|_{B(X)} \lesssim M$,

$$\begin{aligned} \|P_2N_3\|_X &\lesssim M(\|x_y\|_{L^\infty} + \|x_{yy}\|_{L^\infty})\|v\|_X \\ &\lesssim M\mathbb{M}_1(T)\mathbb{M}_2(T)\langle t \rangle^{-5/4} \quad \text{for } t \in [0, T]. \end{aligned}$$

As long as $v(t)$ satisfies the orthogonality condition (5.3) and $\tilde{c}(t, y)$ remains small, we have

$$(9.6) \quad \|v_1(t) - v_2(t)\|_X \lesssim \sup_y |\tilde{c}(t, y)| \|v_1(t)\|_X,$$

and $\frac{1}{2}\|v_2(t)\|_X \leq \|v_1(t)\|_X \leq 2\|v_2(t)\|_X$. Thus we have for $t \in [0, T]$,

$$\begin{aligned} \|v_1(t)\|_X &\lesssim e^{-2b_1\eta_0^2 t} \|v(0)\|_X + \mathbb{M}_1(T) \int_0^t e^{-2b_1\eta_0^2(t-s)} \langle s \rangle^{-3/4} ds \\ &\quad + \mathbb{M}_2(T) \sum_{i=1}^3 \mathbb{M}_i(T) \int_0^t e^{-2b_1\eta_0^2(t-s)} \left\{ 1 + (t-s)^{-3/4} \right\} \langle s \rangle^{-3/4} ds \\ &\lesssim e^{-2b_1\eta_0^2 t} \|v(0)\|_X + \left\{ \mathbb{M}_1(T) + \mathbb{M}_2(T) \sum_{i=1}^3 \mathbb{M}_i(T) \right\} \langle t \rangle^{-3/4}. \end{aligned}$$

Thus we complete the proof. \square

10. VIRIAL ESTIMATES

If we apply the argument in Section 9 to $P_{\geq M}v(t)$, it requires boundedness of $\|v(t)\|_{L^p(\mathbb{R}^2)}$ with $p > 2$, which remains unknown even for small solutions around 0. Instead of the semigroup estimate in Section 3, we will make use of a virial estimates of v in the exponentially weighted space. We remark that the virial estimate for L^2 -solutions to the KP-II equation (2.1) was shown in [6].

Lemma 10.1. *Let $a \in (0, 2)$ and v be a solution to (6.1). There exist positive constants δ_6 , M and C such that if $\sum_{i=1}^3 \mathbb{M}_i(T) < \delta_6$, then*

$$\|v(t)\|_X^2 \leq e^{-2at} \|v(0)\|_X^2 + C \int_0^t e^{-2a(t-s)} (\|\ell(s)\|_X^2 + \|P_{\leq M}v(s)\|_X^2) ds.$$

To prove Lemma 10.1, we use the following.

Claim 10.1. *Let $a > 0$ and $p_n(x) = e^{2an}(1 + \tanh a(x-n))$. There exists a $C > 0$ such that for every $n \in \mathbb{N}$*

(10.1)

$$\left(\int_{\mathbb{R}^2} p_n(x) u^4(x, y) dx dy \right)^{1/2} \leq C \int_{\mathbb{R}^2} p'_n(x) \left((\partial_x u)^2 + (\partial_x^{-1} \partial_y u)^2 + u^2 \right) (x, y) dx dy.$$

Claim 10.1 follows in exactly the same way as [28, Lemma 2] and [26, Claim 5.1]. So we omit the proof.

Proof of Lemma 10.1. Let p_n be as in Claim 10.1. Then $p_n(z) \uparrow e^{2az}$ and $p'_n(z) \uparrow 2ae^{2az}$ as $n \rightarrow \infty$ and $0 < p'_n(z) \leq ap_n(z)$, $|p'''_n(z)| \leq 4a^2 p'_n(z)$ and $ap_n(z)^2 = e^{2az} p'_n(z)$ for $z \in \mathbb{R}$.

First, we will derive a virial identity for $v(t)$ assuming $v_0 \in H^3(\mathbb{R}^2)$ and $\partial_x^{-1} v_0 \in H^2(\mathbb{R}^2)$ so that $v(t) \in C([0, T]; H^3(\mathbb{R}^2))$ and $\partial_x^{-1} v(t) \in C([0, T]; H^2(\mathbb{R}^2))$. Multiplying (6.1) by $2e^{2at} p_n(z) v(t, z, y)$ and integrating the resulting equation by part, we have for $t \in [0, T]$,

(10.2)

$$\begin{aligned} &\frac{d}{dt} \left(e^{2at} \int_{\mathbb{R}^2} p_n(z) v(t, z, y)^2 dz dy \right) + e^{2at} \int_{\mathbb{R}^2} p'_n(z) (\mathcal{E}(v) - 4v^3) (t, z, y) dz dy \\ &= e^{2at} \left\{ \int_{\mathbb{R}^2} 2ap_n(z) v(t, z, y)^2 dz dy + \sum_{k=1}^3 III_k(t) \right\}, \end{aligned}$$

where $\mathcal{E}(v) = 3(\partial_z v)^2 + 3(\partial_z^{-1} \partial_y v)^2 + 4v^2$,

$$\begin{aligned} III_1 &= 2 \int_{\mathbb{R}^2} p_n(z) \ell v(t, z, y) dz dy ds, \\ III_2 &= - \int_{\mathbb{R}^2} p'_n(z) ((\tilde{x}_t(t, y) - 3x_y(t, y)^2) v(t, z, y)^2 dz dy, \\ III_3 &= \int_{\mathbb{R}^2} \left\{ p_n'''(z) + 6[\partial_z, p_n(z)] (\varphi_{c(t, y)}(z) - \psi_{c(t, y), L}(z + 4t)) \right\} v(t, z, y)^2 dz dy. \end{aligned}$$

Integrating (10.2) over $[0, t]$, we have

$$\begin{aligned} (10.3) \quad & e^{2at} \int_{\mathbb{R}^2} p_n(z) v(t, z, y)^2 dz dy + \int_0^t e^{2as} \int_{\mathbb{R}^2} p'_n(z) (\mathcal{E}(v) - 4v^3) (s, z, y) dz dy ds \\ &= \int_{\mathbb{R}^2} p_n(z) v(0, z, y)^2 dz dy + \int_0^t e^{2as} \int_{\mathbb{R}^2} 2ap_n(z) v(s, z, y)^2 dz dy ds \\ &+ \int_0^t e^{2as} \int_{\mathbb{R}^2} \{III_1(s) + III_2(s) + III_3(s)\} ds, \end{aligned}$$

We can prove (10.3) for any $v(t) \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^\infty([0, T]; X)$ satisfying $\sum_{i=1}^3 \mathbb{M}_i(T) < \delta_6$ in the same way as the proof of Lemma 8.1.

By the Schwarz inequality and Claim 10.1,

$$\begin{aligned} (10.4) \quad & \left| \int p'_n(z) v(t, z, y)^3 dz dy \right| \leq \|v(t)\|_{L^2} \left(\int_{\mathbb{R}^2} p'_n(z)^2 v(t, z, y)^4 dz dy \right)^{1/2} \\ & \lesssim \|v(t)\|_{L^2} \int_{\mathbb{R}^2} p'_n(z) \mathcal{E}(v(t, z, y)) dz dy. \end{aligned}$$

By the Schwarz inequality,

$$|III_1| \leq \int p'_n(z) v^2 dz dy + \int \frac{p_n(z)^2}{p'_n(z)} \ell^2 dz dy.$$

Since $Y \subset H^1(\mathbb{R})$, we have $\sup_{t \in [0, T], y \in \mathbb{R}} |\tilde{x}_t(t, y) - 3x_y(t, y)^2| \lesssim \mathbb{M}_1(T)$ from (8.12) and

$$|III_2| \lesssim \mathbb{M}_1(T) \int_{\mathbb{R}^2} p_n(z) v(t, z, y)^2 dz dy.$$

Let

$$M = \sup_{n, z} \frac{|p_n'''(z)|}{p'_n(z)} + 6 \sup_{n, t, y, z} \frac{|[\partial_z, p_n(z)] (\varphi_{c(t, y)}(z) - \psi_{c(t, y), L}(z + 4t))|}{p'_n(z)}.$$

Then

$$|III_3| \leq M \int_{\mathbb{R}^2} p_n(z) v(t, z, y)^2 dz dy.$$

Let $v_< = P_{\leq M} v$ and $v_> = P_{\geq M} v$. For y -high frequencies, the potential term can be absorbed into the left hand side. Indeed it follows from Plancherel's theorem

and the Schwarz inequality that

$$\begin{aligned}
& \int_{\mathbb{R}^2} p'_n(z) ((\partial_z v_>)^2 + (\partial_z^{-1} \partial_y v_>)^2) (t, z, y) dz dy \\
&= \int_{\mathbb{R}^2} p'_n(z) (|\partial_z \mathcal{F}_y(v_>)|^2 + \eta^2 |\partial_z^{-1} \mathcal{F}_y(v_>)|^2) (t, z, \eta) dz d\eta \\
&\geq 2M \int_{\mathbb{R}^2} p'_n(z) v_>(t, z, y)^2 dz dy.
\end{aligned}$$

Combining the above, we have for $t \in [0, T]$,

$$\begin{aligned}
e^{2at} \int_{\mathbb{R}} p_n(z) v(t, z, y)^2 dz dy &\leq \int_{\mathbb{R}} p_n(z) v(0, z, y)^2 dz dy \\
&+ \int_0^t e^{2as} \frac{p_n(z)^2}{p'_n(z)} \ell(s)^2 dz dy ds + M \int_0^t e^{2as} p'_n(z) v_<(s, z, y)^2 dz dy ds
\end{aligned}$$

if δ_6 is sufficiently small. By passing to the limit as $n \rightarrow \infty$, we obtain Lemma 10.1. Thus we complete the proof. \square

Combining Lemmas 9.1 and 10.1, we obtain the following.

Lemma 10.2. *Let a and M be as in Lemmas 9.1 and 10.1. There exist positive constants δ_7 and C such that if $\sum_{i=1}^3 \mathbb{M}_i(T) \leq \delta_7$, then*

$$(10.5) \quad \mathbb{M}_2(T) \leq C(\|v_0\|_X + \mathbb{M}_1(T)).$$

Proof. Since $\chi_M(\eta) = 0$ for $\eta \in \mathbb{R} \setminus [-2M, 2M]$, we have $\|P_{\leq M} v(t)\|_X \leq \|v_1(t)\|_X$. Combining Lemma 10.1 with Lemma 9.1, (9.4) and (9.5), we have for $t \in [0, T]$,

$$\|v(t)\|_X \lesssim e^{-b_2 t} \|v(0)\|_X + \{\mathbb{M}_1(t) + \mathbb{M}_2(T)(\mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_3(T))\} \langle t \rangle^{-3/4}.$$

Since $\|v(0)\|_X \lesssim \|v_0\|_X$ by Lemma 5.2, we obtain (10.5) if δ_7 is sufficiently small. Thus we complete the proof. \square

11. PROOF OF THEOREM 1.1

Now we are in position to complete the proof of Theorem 1.1.

Proof. Since the KP-II equation has the scaling invariance, we may assume that $c_0 = 2$ without loss of generality. Let $\delta_* = \min_{0 \leq i \leq 7} \delta_i/2$.

Since $v_0 \in H^1(\mathbb{R}^2) \cap X$,

$$\tilde{v}(t, x, y) = u(t, x + 4t, y) - \varphi(x) \in C([0, \infty); X \cap H^1(\mathbb{R}^2))$$

(see [29] and Proposition E.1). If $\|v_0\|_X + \|v_0\|_{L^2}$ is sufficiently small, Lemma 5.2 and Remark 5.1 imply that there exists $T > 0$ and $(c(t), x(t))$ satisfying (5.1), (5.3), (5.9) and

$$\|\tilde{c}(t)\|_Y + \|\tilde{x}(t)\|_Y \lesssim \|\tilde{v}(t)\|_X \quad \text{for } t \in [0, T],$$

and it follows that $v(t) \in C([0, T]; X \cap L^2(\mathbb{R}^2))$ and

$$(11.1) \quad \mathbb{M}_{tot}(T) := \mathbb{M}_1(T) + \mathbb{M}_2(T) + \mathbb{M}_3(T) \leq \frac{\delta_*}{2}.$$

By Proposition 5.4, we can extend the decomposition (5.1) satisfying (5.3) beyond $t = T$. Let $T_1 \in (0, \infty]$ be the maximal time such that the decomposition (5.1) with

(5.3) exists for $t \in [0, T_1]$ and $\mathbb{M}_{tot}(T_1) \leq \delta_*$. Suppose $T_1 < \infty$. Then it follows from Lemmas 7.1, 8.2 and 10.2 that

$$(11.2) \quad \mathbb{M}_{tot}(T_1) \lesssim \|v_0\|_{X_1} + \|v_0\|_{L^2(\mathbb{R}^2)} + \mathbb{M}_{tot}(T_1)^2.$$

If $\|v_0\|_{X_1} + \|v_0\|_{L^2(\mathbb{R}^2)}$ is sufficiently small, then $\mathbb{M}_{tot}(T_1) \leq \delta_*/2$ follows from (11.2), which contradicts to the definition of T_1 . Thus we prove $T_1 = \infty$ and

$$(11.3) \quad \mathbb{M}_{tot}(\infty) \lesssim \|v_0\|_{X_1} + \|v_0\|_{L^2}.$$

Now we will prove (1.5) and (1.8). By (5.1), (8.11) and (11.3),

$$\begin{aligned} \|u(t, x, y) - \varphi_{c(t, y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} &\leq \|v(t)\|_{L^2(\mathbb{R}^2)} + \|\tilde{\psi}_{c(t, y)}\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \mathbb{M}_3(\infty) + \mathbb{M}_1(\infty), \end{aligned}$$

$$\begin{aligned} \|e^{ax}(u(t, x + x(t, y), y) - \varphi_{c(t, y)}(x))\|_{L^2} &\leq \|v(t)\|_X + \|\tilde{\psi}_{c(t, y)}\|_X \\ &\lesssim \mathbb{M}_2(\infty)\langle t \rangle^{-3/4} + \mathbb{M}_1(\infty)e^{-a(4t+L)}\langle t \rangle^{-1/4}. \end{aligned}$$

Since $\|f\|_{L^\infty} \lesssim \|f\|_Y^{1/2}\|\partial_y f\|_Y^{1/2}$ for any $f \in Y$, we see that (1.6) and (1.7) follow immediately from (11.3) and (8.12). Thus we complete the proof of Theorem 1.1. \square

12. PROOF OF THEOREM 1.2

In this section, we will prove orbital instability of line solitons. For the purpose, we will utilize that (b, x_y) is a solution to the diffusion wave equation (7.2) and its profile can be approximated by the heat kernel in some region.

Proof of Theorem 1.2. First we remark that if $\|u(t, x) - \varphi_{c_0}(x - x_0)\|_{L^2(\mathbb{R}^2)} < \infty$, then $x_0 = 2c_0t$. Indeed, it follow from [29] that $u(t, x, y) - \varphi_{c_0}(x - 2c_0t) \in L^2(\mathbb{R}^2)$ for every $t \geq 0$ and

$$\begin{aligned} &\|\varphi_{c_0}(x - 2c_0t) - \varphi_{c_0}(x - x_0)\|_{L^2(\mathbb{R}^2)} \\ &\leq \|u(t, x, y) - \varphi_{c_0}(x - x_0)\|_{L^2(\mathbb{R}^2)} + \|u(t, x, y) - \varphi_{c_0}(x - 2c_0t)\|_{L^2(\mathbb{R}^2)} < \infty, \end{aligned}$$

whereas $\|\varphi_{c_0}(\cdot - 2c_0t) - \varphi_{c_0}(\cdot - x_0)\|_{L^2(\mathbb{R}^2)} = \infty$ if $x_0 \neq 2c_0t$.

On the other hand, Theorem 1.1 implies that

$$\begin{aligned} &\|u(t, \cdot) - \varphi_{c_0}(x - 2c_0t)\|_{L^2(\mathbb{R}^2)} \\ &\geq \|\varphi_{c_0}(x - x(t, y)) - \varphi_{c_0}(x - 2c_0t)\|_{L^2(\mathbb{R}^2)} - \|u(t, x, y) - \varphi_{c(t, y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \\ &\quad - \|\varphi_{c(t, y)}(x - x(t, y)) - \varphi_{c_0}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \\ &\gtrsim \|x(t, y) - 2c_0t\|_{L^2(\mathbb{R})} - O(\varepsilon). \end{aligned}$$

Thus to prove orbital instability of line solitons, it suffices to show that $\|x(t, \cdot)\|_{L^2(\mathbb{R})}$ grows up as $t \rightarrow \infty$.

Now we will construct a solution satisfying $\|x(t, \cdot)\|_Y \gtrsim t^{1/4}$ as $t \rightarrow \infty$. We may assume that $c_0 = 2$ without loss of generality. If $b(0)$ and $x_y(0)$ are sufficiently small and $\int_{\mathbb{R}} b(0) dy$ is nonzero, then $e^{tA_0}(b(0), x_y(0))$ is expected to be the main part of the solution $(b(t), x_y(t))$. To investigate the behavior of $e^{tA_0}(b(0), x_y(0))$, we represent the semigroup e^{tA_0} by using the heat kernel $H_t(y)$. Let

$$A_{0,1}(\eta) = \begin{pmatrix} -3\eta^2 & 8i\eta \\ i\eta(2 + \frac{1}{8}\eta^2) & -\eta^2 \end{pmatrix}, \quad A_{0,2}(\eta) = \frac{1}{\eta^3} (A_0(\eta) - A_{0,1}(\eta)).$$

Note that $A_{0,1}$ is equal to A_* in Lemma 4.2 and that $A_{0,2} = \begin{pmatrix} O(\eta) & O(1) \\ O(1) & O(\eta) \end{pmatrix}$.

Let $U_1(t, s)$ be the 2×2 matrix such that

$$\partial_t U_1(t) = A_1(t, 0)U_1(t), \quad \lim_{t \rightarrow \infty} U_1(t) = I.$$

Since $|A_1(t, 0)| \lesssim e^{-a(4t+L)}$ for $t \geq 0$, we have $\sup_{\tau \geq t} |U_1(\tau) - I| \lesssim e^{-a(4t+L)}$. Now let

$$\begin{pmatrix} b(t) \\ x_y(t) \end{pmatrix} = U_1(t) \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.$$

Then

$$(12.1) \quad \partial_t \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} = (A_0(D_y) + D_y A_2(t, D_y)) \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} + \sum_{i=1}^8 U_1(t)^{-1} \text{diag}(1, \partial_y) \mathcal{N}_i,$$

where $A_2(t, \eta) = A_{21}(t, \eta) + A_{22}(t, \eta)$ and

$$A_{21}(t, \eta) = \frac{U_1(t)^{-1} A_0(\eta) U_1(t) - A_0(\eta)}{\eta},$$

$$A_{22}(t, \eta) = U_1(t)^{-1} \frac{A_1(t, \eta) - A_1(t, 0)}{\eta} U_1(t).$$

Clearly, we have $\|A_2(t, D_y)\|_{B(Y)} \lesssim e^{-a(4t+L)}$. By the variation of constants formula,

$$\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} = e^{tA_{0,1}} \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} + IV_1 + IV_2 + IV_3,$$

where

$$IV_1 = \int_0^t e^{(t-s)A_{0,1}} D_y^3 A_{0,2} \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix} ds,$$

$$IV_2 = \int_0^t e^{(t-s)A_{0,1}} D_y A_2(s, D_y) \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix} ds,$$

$$IV_3 = \sum_{i=1}^8 \int_0^t e^{(t-s)A_{0,1}} U_1(s)^{-1} \text{diag}(1, \partial_y) \mathcal{N}_i(s) ds.$$

Let $h \in C_0^\infty(-\eta_0, \eta_0)$ such that $h(0) = 1$ and let

$$u(0, x, y) = \varphi_{2+c_*(y)}(x) - \psi_{2+c_*(y), L}(x), \quad c_*(y) = 2 + \varepsilon(\mathcal{F}_\eta^{-1}h)(y).$$

Then it follows from Lemma 5.2 that $\tilde{c}(0, y) = c_*(y)$, $x(0, y) \equiv 0$ and $v(0, \cdot) = 0$. Since $\|b(0) - \tilde{c}(0)\|_{Y_1} \lesssim \|\tilde{c}(0)\|_Y^2$ by Claim D.6 and $\|U_1(t) - I\|_{B(Y)} \lesssim e^{-a(4t+L)}$,

$$(12.2) \quad \left\| \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} - \begin{pmatrix} c_* \\ 0 \end{pmatrix} \right\|_{Y_1} = \left\| U_1(0)^{-1} \begin{pmatrix} b(0) \\ 0 \end{pmatrix} - \begin{pmatrix} \tilde{c}(0) \\ 0 \end{pmatrix} \right\|_{Y_1} \lesssim \varepsilon(\varepsilon + e^{-aL}).$$

Since $\|e^{tA_{0,1}}\|_{B(Y_1, Y)} \lesssim (1+t)^{-1/4}$ by Lemma 4.2, it follows from (12.2) that

$$\left\| e^{tA_{0,1}} \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} - e^{tA_{0,1}} \begin{pmatrix} c_* \\ 0 \end{pmatrix} \right\|_Y \lesssim \varepsilon(\varepsilon + e^{-aL})(1+t)^{-1/4}.$$

By Corollary 4.3,

$$\left\| e^{tA_{0,1}} \begin{pmatrix} c_* \\ 0 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 2(e^{4t\partial_y}(H_{2t} * c_*) + e^{-4t\partial_y}(H_{2t} * c_*)) \\ e^{4t\partial_y}(H_{2t} * c_*) - e^{-4t\partial_y}(H_{2t} * c_*) \end{pmatrix} \right\|_Y \lesssim \varepsilon \langle t \rangle^{-3/4}.$$

Since $c_* = \varepsilon \mathcal{F}^{-1}h$ and $h(0) = 1$, it follows from Plancherel's theorem that

$$\begin{aligned} \left\| \tilde{P}_1 \{ e^{\pm 4t\partial_y} (H_{2t} * c_*) - \varepsilon e^{\pm 4t\partial_y} H_{2t} \} \right\|_Y &= \varepsilon \left\| e^{-2t\eta^2} (h(\eta) - h(0)) \right\|_{L^2(-\eta_0, \eta_0)} \\ &\lesssim \varepsilon \sup_{\eta} |h'(\eta)| \left\| \eta e^{-2t\eta^2} \right\|_{L^2(-\eta_0, \eta_0)} \lesssim \varepsilon \langle t \rangle^{-3/4}. \end{aligned}$$

Thus we have

$$\begin{aligned} (12.3) \quad & \left\| e^{tA_{0,1}} \begin{pmatrix} b_1(0) \\ b_2(0) \end{pmatrix} - \frac{\varepsilon}{4} \tilde{P}_1 \begin{pmatrix} 2(e^{4t\partial_y} H_{2t} + e^{-4t\partial_y} H_{2t}) \\ e^{4t\partial_y} H_{2t} - e^{-4t\partial_y} H_{2t} \end{pmatrix} \right\|_Y \\ & \lesssim \varepsilon (\varepsilon + e^{-aL}) \langle t \rangle^{-1/4} + \varepsilon \langle t \rangle^{-3/4}. \end{aligned}$$

In view of the proof of Theorem 1.1, we have for $k = 0$ and 1,

$$\sup_{t \geq 0} \langle t \rangle^{(2k+1)/4} (\|b_1(t)\|_Y + \|b_2(t)\|_Y) \lesssim \mathbb{M}_1(\infty) \lesssim \varepsilon.$$

By Lemma 4.2 and Claim 4.1,

$$\begin{aligned} (12.4) \quad & \|IV_1\|_Y \lesssim \int_0^t \|\partial_y^2 e^{(t-s)A_{0,1}}\|_{B(Y)} (\|\partial_y b_1(s)\|_Y + \|\partial_y b_2(s)\|_Y) ds \\ & \lesssim \mathbb{M}_1(t) \int_0^t \langle t-s \rangle^{-1} \langle s \rangle^{-3/4} ds \lesssim \varepsilon \langle t \rangle^{-3/4} \log \langle t \rangle, \end{aligned}$$

and

$$\begin{aligned} (12.5) \quad & \|IV_2\|_Y \lesssim \int_0^t \|\partial_y e^{(t-s)A_{0,1}}\|_{B(Y)} \|A_2(s, D_y)\|_{B(Y)} (\|b_1(s)\|_Y + \|b_2(s)\|_Y) ds \\ & \lesssim \mathbb{M}_1(t) \int_0^t \langle t-s \rangle^{-1/2} e^{-a(4s+L)} \langle s \rangle^{-1/4} ds \lesssim \varepsilon \langle t \rangle^{-1/2}. \end{aligned}$$

Using Lemma 4.2, we can prove

$$(12.6) \quad \|IV_3\|_Y \lesssim \langle t \rangle^{-1/4} (\mathbb{M}_1(\infty)^2 + \mathbb{M}_2(\infty)^2) \lesssim \varepsilon^2 \langle t \rangle^{-1/4}$$

in exactly the same way as the proof of Lemma 7.1. Combining (12.3)–(12.6), we have

$$\left\| \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} - \frac{\varepsilon}{4} \tilde{P}_1 \begin{pmatrix} 2(e^{4t\partial_y} H_{2t} + e^{-4t\partial_y} H_{2t}) \\ e^{4t\partial_y} H_{2t} - e^{-4t\partial_y} H_{2t} \end{pmatrix} \right\|_Y \lesssim \varepsilon (\varepsilon + e^{-aL}) \langle t \rangle^{-1/4} + \varepsilon \langle t \rangle^{-1/2}.$$

Since $|U_1(t) - I| \lesssim e^{-a(4t+L)}$ and $\|(I - \tilde{P}_1)H_{2t}\|_{L^2} \lesssim e^{-2t\eta_0^2}$,

$$(12.7) \quad \left\| x_y(t) - \frac{\varepsilon}{4} (e^{4t\partial_y} H_{2t} - e^{-4t\partial_y} H_{2t}) \right\|_{L^2} \lesssim \varepsilon (\varepsilon + e^{-aL}) \langle t \rangle^{-1/4} + \varepsilon \langle t \rangle^{-1/2}.$$

Now let d_1 and d_2 be constants satisfying $d_2 > d_1 > 1$ and let $y_1 \in [-4t + 4(d_1 - 1)\sqrt{t}, -4t + 4d_1\sqrt{t}]$, $y_2 \in [-4t + 4d_2\sqrt{t}, -4t + 4(d_2 + 1)\sqrt{t}]$ for $t \geq 0$. By (12.7),

there exist positive constants C_1 , C_2 and t_1 such that

$$\begin{aligned}
x(t, y_2) - x(t, y_1) &= \int_{y_1}^{y_2} x_y(t, \tilde{y}) d\tilde{y} \\
&\geq \frac{\varepsilon}{4} \int_{y_1}^{y_2} (H_{2t}(\tilde{y} + 4t) - H_{2t}(\tilde{y} - 4t)) d\tilde{y} \\
&\quad - (y_2 - y_1)^{1/2} \left\| x_y(t) - \frac{\varepsilon}{4} (e^{4t\partial_y} H_{2t} - e^{-4t\partial_y} H_{2t}) \right\|_{L^2} \\
&\gtrsim \varepsilon \int_{d_1\sqrt{t}}^{d_2\sqrt{t}} H_{2t}(y) dy - C_1 \varepsilon (\varepsilon + e^{-aL} + \langle t \rangle^{-1/4}) \\
&= \varepsilon \left(\operatorname{erf}(\sqrt{2}d_2) - \operatorname{erf}(\sqrt{2}d_1) \right) - C_1 \varepsilon (\varepsilon + e^{-aL} + \langle t \rangle^{-1/4}), \\
&\geq C_2 \varepsilon \quad \text{for } t \geq t_1,
\end{aligned}$$

if ε and e^{-aL} are sufficiently small. Recall that $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x_1^2} dx_1$. Since L is an auxiliary parameter introduced in Section 5 which can be chosen arbitrary large, we see that $|x(t, y)| \geq C_2 \varepsilon / 2$ either on $[-4t + 4(d_1 - 1)\sqrt{t}, -4t + 4d_1\sqrt{t}]$ or on $[-4t + 4d_2\sqrt{t}, -4t + 4(d_2 + 1)\sqrt{t}]$. Therefore $\|x(t)\|_Y \gtrsim \varepsilon \langle t \rangle^{1/4}$. Thus we complete the proof. \square

13. PROOF OF THEOREM 1.3

In order to prove Theorem 1.3, we will show that the first order asymptotics of solutions to (12.1) around $y = \pm 4t + O(\sqrt{t})$ is given by a sum of self-similar solutions to the Burgers equations. We apply the scaling argument by Karch ([20]) to obtain the asymptotics of (12.1) and use a virial type estimate to show that interaction between $b_1(t, y)$ and $b_2(t, y)$ tends to 0 around $y = \pm 4t + O(\sqrt{t})$ as $t \rightarrow \infty$. Since $\sup_{t>0} t^{1/4} (\|b_1(t)\|_{L^2(\mathbb{R})} + \|b_2(t)\|_{L^2(\mathbb{R})}) \ll 1$, we have the uniqueness of the limiting profile.

Roughly speaking, a solution of (12.1) can be decomposed into two parts that move to the opposite direction. Now we recenter each component of solutions to (12.1) and diagonalize the equations. Let $A_*(\eta)$, $\Pi_*(\eta)$ and $\omega(\eta)$ be as (4.6) with $\mu = \mu_3$. By the change of variables

$$\begin{aligned}
\mathbf{b}(t, y) &= \begin{pmatrix} b_1(t, y) \\ b_2(t, y) \end{pmatrix}, \quad \Pi_1(t, \eta) = \frac{1}{4i} \Pi_*(\eta) \operatorname{diag}(e^{4it\eta}, e^{-4it\eta}), \\
\mathbf{d}(t, y) &= \mathcal{F}_\eta^{-1} \Pi_1(t, \eta)^{-1} (\mathcal{F}_y \mathbf{b})(t, \eta),
\end{aligned}$$

we have

$$(13.1) \quad \partial_t \mathbf{d} = \{2\partial_y^2 I + \partial_y(A_3(t, D_y) + A_4(t, D_y))\} \mathbf{d} + A_5(t, D_y) \sum_{i=1}^8 \operatorname{diag}(1, \partial_y) \mathcal{N}_i,$$

where

$$\begin{aligned}
A_3(t, \eta) &= (\omega(\eta) - 4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\eta^{-1} \Pi_1(t, \eta)^{-1} (A_0(\eta) - A_*(\eta)), \\
A_4(t, \eta) &= -i\Pi_1(t, \eta)^{-1} A_2(t, \eta) \Pi_1(t, \eta), \quad A_5(t, \eta) = \Pi_1(t, \eta)^{-1} U_1(t)^{-1}.
\end{aligned}$$

To detect the dominant part of the equation, let us consider the rescaled solution $\mathbf{d}_\lambda(t, y) = \lambda \mathbf{d}(\lambda^2 t, \lambda y)$. Our aim is to find a self-similar profile $\mathbf{d}_\infty(t, y)$ such that

$$(13.2) \quad \lambda \mathbf{d}_\infty(\lambda^2 t, \lambda y) = \mathbf{d}_\infty(t, y),$$

and that for any t_1 and t_2 satisfying $0 < t_1 \leq t_2 < \infty$ and any $R > 0$,

$$(13.3) \quad \lim_{\lambda \rightarrow \infty} \sup_{t \in [t_1, t_2]} \|\mathbf{d}_\lambda(t, y) - \mathbf{d}_\infty(t, y)\|_{L^2(|y| < R)} = 0.$$

If (13.2) and (13.3) hold, then letting $\lambda = t^{1/2} \rightarrow \infty$, we have

$$(13.4) \quad \begin{aligned} t^{1/4} \|\mathbf{d}(t, \cdot) - \mathbf{d}_\infty(t, \cdot)\|_{L^2(|y| < R\sqrt{t})} &= \lambda^{1/2} \|\mathbf{d}(\lambda^2, \cdot) - \mathbf{d}_\infty(\lambda^2, \cdot)\|_{L^2(|y| < \lambda R)} \\ &= \|\mathbf{d}_\lambda(1, \cdot) - \mathbf{d}_\infty(1, \cdot)\|_{L^2(|y| < R)} \rightarrow 0. \end{aligned}$$

To prove (13.3), we need the upper bounds of $\mathbf{d}_\lambda(t)$ for $k = 0, 1$ that do not depend on $\lambda \geq 1$.

Lemma 13.1. *Let ε be as in Theorem 1.1. Then there exists a positive constants C such that for any $\lambda \geq 1$ and $t \in (0, \infty)$,*

$$(13.5) \quad \sum_{k=0,1} \|\partial_y^k \mathbf{d}_\lambda(t, \cdot)\|_{L^2} \leq C\varepsilon t^{-(2k+1)/4}, \quad \|\partial_y^2 \mathbf{d}_\lambda(t, \cdot)\|_{L^2} \leq C\varepsilon \lambda^{1/2} t^{-1},$$

$$(13.6) \quad \|\partial_t \mathbf{d}_\lambda(t, \cdot)\|_{H^{-2}} \leq C(t^{-1/4} + t^{-3/2})\varepsilon.$$

Proof. Since $\mathbb{M}_1(\infty) \lesssim \varepsilon$ by (11.3), we have

$$\sum_{k=0,1} \sup_{t \geq 0} \langle t \rangle^{(2k+1)/4} \|\partial_y^k \mathbf{d}(t)\|_Y + \langle t \rangle \|\partial_y^2 \mathbf{d}(t)\|_Y \lesssim \varepsilon.$$

Thus we have

$$\begin{aligned} \|\partial_y^k \mathbf{d}_\lambda(t, \cdot)\|_{L^2} &= \lambda^{(2k+1)/2} \|\partial_y^k \mathbf{d}(\lambda^2 t, \cdot)\|_Y \\ &\lesssim \lambda^{(2k+1)/2} (1 + \lambda^2 t)^{-(2k+1)/4} \varepsilon \lesssim t^{-(2k+1)/4} \varepsilon \quad \text{for } k = 0, 1, \\ \|\partial_y^2 \mathbf{d}_\lambda(t, \cdot)\|_{L^2} &\lesssim \lambda^{5/2} \|\partial_y^2 \mathbf{d}(\lambda^2 t, \cdot)\|_Y \lesssim \lambda^{5/2} (1 + \lambda^2 t)^{-1} \varepsilon \lesssim \lambda^{1/2} t^{-1} \varepsilon. \end{aligned}$$

Thus we prove (13.5).

Next we will show (13.6). Let $\mathcal{N}'(t, y) + \partial_y \mathcal{N}''(t, y) = \text{diag}(1, \partial_y) \sum_{i=2}^8 \mathcal{N}_i(t, y)$,

$$\mathcal{N}'_\lambda(t, y) = \lambda^3 \mathcal{N}'(\lambda^2 t, \lambda y), \quad \mathcal{N}''_\lambda(t, y) = \lambda^2 \mathcal{N}''(\lambda^2 t, \lambda y),$$

$$\tilde{\mathcal{N}}(t, y) = \tilde{P}_1 \begin{pmatrix} n_1(t, y) \\ n_2(t, y) \end{pmatrix}, \quad \tilde{\mathcal{N}}_\lambda(t, y) = \lambda^2 \tilde{\mathcal{N}}(\lambda^2 t, \lambda y).$$

Then (13.1) can be rewritten as

$$(13.7) \quad \begin{aligned} \partial_t \mathbf{d}_\lambda &= \{2\partial_y^2 I + \lambda \partial_y (A_3(\lambda^2 t, \lambda^{-1} D_y) + A_4(\lambda^2 t, \lambda^{-1} D_y))\} \mathbf{d}_\lambda \\ &\quad + A_5(\lambda^2 t, \lambda^{-1} D_y) \{\partial_y (\tilde{\mathcal{N}}_\lambda + \mathcal{N}''_\lambda) + \mathcal{N}'_\lambda\}. \end{aligned}$$

By (13.7),

$$(13.8) \quad \begin{aligned} \|\partial_t \mathbf{d}_\lambda\|_{H^{-2}} &\leq 2\|\mathbf{d}_\lambda\|_{L^2} + \lambda \|A_3(\lambda^2 t, \lambda^{-1} D_y) \mathbf{d}_\lambda\|_{H^{-1}} + \lambda \|A_4(\lambda^2 t, \lambda^{-1} D_y) \mathbf{d}_\lambda\|_{L^2} \\ &\quad + \|A_5(\lambda^2 t, \lambda^{-1} D_y) (\tilde{\mathcal{N}}_\lambda + \mathcal{N}''_\lambda)\|_{L^2} + \|A_5(\lambda^2 t, \lambda^{-1} D_y) \mathcal{N}'_\lambda\|_{L^2} \end{aligned}$$

Now we will estimate each term of the right hand side. By (7.3) and the fact that $\omega(\eta) = 4 + O(\eta^2)$, we have

$$(13.9) \quad |A_3(\lambda^2 t, \lambda^{-1} \eta)| \lesssim \lambda^{-2} \eta^2.$$

Thus by (13.5) and Plancherel's theorem,

$$(13.10) \quad \begin{aligned} \lambda \|A_3(\lambda^2 t, \lambda^{-1} D_y) \mathbf{d}_\lambda\|_{H^{-1}} &\lesssim \lambda^{-1} \|\langle \eta \rangle^{-1} \eta^2 (\mathcal{F}_y \mathbf{d}_\lambda)(t, \eta)\|_{L^2} \\ &\lesssim \lambda^{-1} \|\partial_y \mathbf{d}_\lambda(t, \cdot)\|_{L^2} \lesssim \lambda^{-1} t^{-3/4} \varepsilon. \end{aligned}$$

Since $\|A_4(t, D_y)\|_{B(Y)} \lesssim \|A_2(t, D_y)\|_{B(Y)} \lesssim e^{-a(4t+L)}$, it follows from (13.5) and the scaling argument that

$$(13.11) \quad \begin{aligned} \lambda \|A_4(\lambda^2 t, \lambda^{-1} D_y) \mathbf{d}_\lambda(t, \cdot)\|_{L^2} &= \lambda^{3/2} \|A_4(\lambda^2 t, D_y) \mathbf{d}(\lambda^2 t, \cdot)\|_Y \\ &\lesssim \lambda^{3/2} e^{-a(4\lambda^2 t+L)} \|\mathbf{d}(\lambda^2 t, \cdot)\|_Y \\ &\lesssim \lambda t^{-1/4} e^{-a(4\lambda^2 t+L)} \varepsilon \lesssim \lambda^{-1/4} t^{-7/8} \varepsilon. \end{aligned}$$

Following the proof of Lemma 7.1, we have for $t \geq 0$,

$$(13.12) \quad \|\tilde{\mathcal{N}}\|_Y \lesssim \langle t \rangle^{-3/4} \varepsilon^2, \quad \|\mathcal{N}'\|_Y \lesssim \langle t \rangle^{-3/2} \varepsilon^2, \quad \|\mathcal{N}''\|_Y \lesssim \langle t \rangle^{-5/4} \varepsilon^2.$$

Nonlinear terms decay $t^{-1/4}$ times faster in (13.12) than those in (7.6) and (7.12) because Y and Y_1 have the same scaling as $L^2(\mathbb{R})$ and $L^1(\mathbb{R})$, respectively. By (13.12),

$$\begin{aligned} \|\tilde{\mathcal{N}}_\lambda\|_{L^2} &= \lambda^{3/2} \|\tilde{\mathcal{N}}(\lambda^2 t, \cdot)\|_Y \lesssim \lambda^{3/2} (1 + \lambda^2 t)^{-3/4} \varepsilon^2 \lesssim t^{-3/4} \varepsilon^2, \\ \|\mathcal{N}'_\lambda\|_{L^2} &= \lambda^{5/2} \|\mathcal{N}'(\lambda^2 t, \cdot)\|_Y \lesssim \lambda^{5/2} (1 + \lambda^2 t)^{-3/2} \varepsilon^2 \lesssim \lambda^{-1/2} t^{-3/2} \varepsilon^2, \\ \|\mathcal{N}''_\lambda\|_{L^2} &= \lambda^{3/2} \|\mathcal{N}''(\lambda^2 t, \cdot)\|_{L^2} \lesssim \lambda^{3/2} (1 + \lambda^2 t)^{-5/4} \varepsilon^2 \lesssim \lambda^{-1/4} t^{-7/8} \varepsilon^2. \end{aligned}$$

Since $\sup_{\lambda \geq 1} \|A_5(\lambda^2 t, \lambda^{-1} D_y)\|_{B(L^2)} \lesssim 1$, we have

$$(13.13) \quad \|A_5(\lambda^2 t, \lambda^{-1} D_y) \tilde{\mathcal{N}}_\lambda\|_{L^2} \lesssim t^{-3/4} \varepsilon^2,$$

$$(13.14) \quad \|A_5(\lambda^2 t, \lambda^{-1} D_y) \mathcal{N}''_\lambda\|_{L^2} \lesssim \lambda^{-1/4} t^{-7/8} \varepsilon^2,$$

$$(13.15) \quad \|A_5(\lambda^2 t, \lambda^{-1} D_y) \mathcal{N}'_\lambda\|_{L^2} \lesssim \lambda^{-1/2} t^{-3/2} \varepsilon^2.$$

Combining (13.5) and (13.8)–(13.11), (13.13)–(13.15), we obtain (13.6). \square

By Lemma 13.1 and the Arzelà-Ascoli theorem, we have the following.

Corollary 13.2. *There exist a sequence $\{\lambda_n\}_{n \geq 1}$ satisfying $\lim_{n \rightarrow \infty} \lambda_n = \infty$ and $\mathbf{d}_\infty(t, y)$ such that*

$$\begin{aligned} \mathbf{d}_{\lambda_n}(t, \cdot) &\rightarrow \mathbf{d}_\infty(t, \cdot) \quad \text{weakly star in } L_{loc}^\infty((0, \infty); H^1(\mathbb{R})), \\ \partial_t \mathbf{d}_{\lambda_n}(t, \cdot) &\rightarrow \partial_t \mathbf{d}_\infty(t, \cdot) \quad \text{weakly star in } L_{loc}^\infty((0, \infty); H^{-2}(\mathbb{R})), \\ \sup_{t > 0} t^{1/4} \|\mathbf{d}_\infty(t)\|_{L^2} &\leq C \varepsilon, \end{aligned}$$

where C is a constant given in Lemma 13.1. Moreover, for any $R > 0$ and t_1, t_2 with $0 < t_1 \leq t_2 < \infty$,

$$\lim_{n \rightarrow \infty} \sup_{t \in [t_1, t_2]} \|\mathbf{d}_{\lambda_n}(t, \cdot) - \mathbf{d}_\infty(t, \cdot)\|_{L^2(|y| \leq R)} = 0.$$

Next we will show that $\mathbf{d}_\infty(t)$ is a self-similar solution to a system of Burgers equations. To begin with, we will prove the following.

Lemma 13.3. *Let $\mathbf{d}_\infty(t) = {}^t(d_+(t, y), d_-(t, y))$. Then for $t > 0$ and $y \in \mathbb{R}$,*

$$(13.16) \quad \begin{cases} \partial_t d_+ = 2\partial_y^2 d_+ + 4\partial_y(d_+^2), \\ \partial_t d_- = 2\partial_y^2 d_- - 4\partial_y(d_-^2), \end{cases}$$

and

$$(13.17) \quad \lim_{t \downarrow 0} \int_{\mathbb{R}} \mathbf{d}_{\infty}(t, y) h(y) dy = \sqrt{2\pi}(\mathcal{F}_y \tilde{\mathbf{d}})(0, 0) h(0) \quad \text{for any } h \in H^2(\mathbb{R}),$$

where

$$\tilde{\mathbf{d}}(t, y) = \mathbf{d}(t, y) + \int_t^{\infty} A_5(s, D_y) \mathcal{N}'(s, y) ds.$$

Proof. Let $\tilde{\mathbf{d}}_{\lambda}(t, y) = \lambda \tilde{\mathbf{d}}(\lambda t, \lambda^2 y)$. The limiting profile of $\mathbf{d}_{\lambda}(t)$ and $\tilde{\mathbf{d}}_{\lambda}(t)$ as $\lambda \rightarrow \infty$ are the same for every $t > 0$. Indeed, it follows from (13.15) that

$$(13.18) \quad \begin{aligned} \|\tilde{\mathbf{d}}_{\lambda}(t, \cdot) - \mathbf{d}_{\lambda}(t, \cdot)\|_{L^2} &\lesssim \int_t^{\infty} \|A_5(\lambda^2 s, \lambda^{-1} D_y) \mathcal{N}'_{\lambda}(s, y)\|_{L^2} ds \\ &\lesssim \lambda^{-1/2} \int_t^{\infty} \tau^{-3/2} d\tau \lesssim \lambda^{-1/2} t^{-1/2}. \end{aligned}$$

By (13.7),

$$(13.19) \quad \begin{aligned} \partial_t \tilde{\mathbf{d}}_{\lambda} &= 2\partial_y^2 \mathbf{d}_{\lambda} + \lambda \partial_y \{ (A_3(\lambda^2 t, \lambda^{-1} D_y) + A_4(\lambda^2 t, \lambda^{-1} D_y)) \} \mathbf{d}_{\lambda} \\ &\quad + \partial_y A_5(\lambda^2 t, \lambda^{-1} D_y) (\tilde{\mathcal{N}}_{\lambda} + \mathcal{N}_{\lambda}''), \end{aligned}$$

and we have $\sup_{\lambda \geq 1} \|\partial_t \tilde{\mathbf{d}}_{\lambda}(t, \cdot)\|_{H^{-2}} \lesssim t^{-1/4} + t^{-7/8}$ from (13.5), (13.10), (13.11), (13.13), (13.14) and (13.19). Thus for $t > s > 0$ and $h \in H^2(\mathbb{R})$,

$$\left| \int_{\mathbb{R}} \tilde{\mathbf{d}}_{\lambda}(t, y) h(y) dy - \int_{\mathbb{R}} \tilde{\mathbf{d}}_{\lambda}(s, y) h(y) dy \right| \leq C \{ (t-s)^{3/4} + (t-s)^{1/8} \},$$

where C is a constant independent of λ . Passing to the limit as $s \downarrow 0$ in the above, we obtain for $t > 0$,

$$(13.20) \quad \left| \int_{\mathbb{R}} \tilde{\mathbf{d}}_{\lambda}(t, y) h(y) dy - \int_{\mathbb{R}} \tilde{\mathbf{d}}_{\lambda}(0, y) h(y) dy \right| \leq C(t^{3/4} + t^{1/8}).$$

Since $\mathbf{d}(0, \cdot) \in Y_1$ by the definition and $\|\mathcal{N}'(\tau, \cdot)\|_{Y_1} \lesssim \langle \tau \rangle^{-5/4}$, we have

$$\tilde{\mathbf{d}}_1(0, y) = \mathbf{d}(0, y) + \int_0^{\infty} A_5(\tau, D_y) \mathcal{N}'(\tau, y) d\tau \in Y_1,$$

and it follows from Lebesgue's dominated convergence theorem that

$$\int_{\mathbb{R}} \tilde{\mathbf{d}}_{\lambda}(0, y) h(y) dy = \int_{\mathbb{R}} (\mathcal{F}_y \tilde{\mathbf{d}}_1)(0, \lambda^{-1} \eta) (\mathcal{F}_y^{-1} h)(\eta) d\eta \rightarrow \sqrt{2\pi}(\mathcal{F}_y \tilde{\mathbf{d}}_1)(0, 0) h(0)$$

as $\lambda \rightarrow \infty$ for any $h \in H^1(\mathbb{R})$. Letting $\lambda = \lambda_n$ and passing to the limit as $n \rightarrow \infty$ in (13.20), we see that (13.17) follows from Corollary 13.2.

Next, we will show (13.16). By Corollary 13.2 and (13.18),

$$(13.21) \quad \partial_t \tilde{\mathbf{d}}_{\lambda_n} - 2\partial_y^2 \mathbf{d}_{\lambda_n} \rightarrow \partial_t \mathbf{d}_{\infty} - 2\partial_y^2 \mathbf{d}_{\infty}.$$

By (13.10), (13.11) and (13.14),

$$(13.22) \quad \lambda \{ (A_3(\lambda_n^2 t, \lambda_n^{-1} D_y) + A_4(\lambda_n^2 t, \lambda_n^{-1} D_y)) \} \mathbf{d}_{\lambda_n} + A_5(\lambda_n^2 t, \lambda^{-1} D_y) \mathcal{N}_{\lambda_n}'' \rightarrow 0,$$

as $n \rightarrow \infty$ in $\mathcal{D}'((0, \infty) \times \mathbb{R})$.

Now we investigate the limit of $\tilde{\mathcal{N}}_{\lambda}$. Let

$$\mathbf{d}_{\lambda}(t, y) = \begin{pmatrix} d_{+, \lambda}(t, y) \\ d_{-, \lambda}(t, y) \end{pmatrix}.$$

By the definition of $\mathbf{d}(t)$,

$$\begin{pmatrix} b(t, \cdot) \\ x_y(t, \cdot) \end{pmatrix} = U_1(t) \Pi_1(t, D_y) \mathbf{d}(t, \cdot).$$

Since $|U_1(t) - I| \lesssim e^{-a(4t+L)}$ and

$$(13.23) \quad \left| (4i)^{-1} \Pi_*(\lambda^{-1} \eta) - \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \right| \lesssim \lambda^{-1} \eta \text{quad for } \lambda \geq 1,$$

we have

$$(13.24) \quad \left\| \lambda \begin{pmatrix} b(\lambda^2 t, \lambda \cdot) \\ x_y(\lambda^2 t, \lambda \cdot) \end{pmatrix} + \begin{pmatrix} 8 & 8 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} e^{4\lambda t \partial_y} d_{+, \lambda}(t, \cdot) \\ e^{-4\lambda t \partial_y} d_{-, \lambda}(t, \cdot) \end{pmatrix} \right\|_{L^2} \\ \lesssim (\lambda^{-1} + e^{-4a\lambda^2 t}) \|\mathbf{d}_\lambda(t)\|_{H^1} \lesssim (\lambda^{-1} + e^{-4a\lambda^2 t}) (t^{-1/4} + t^{-3/4}).$$

Recall that $(n_1, n_2) = (6bx_y, 2(\tilde{c} - b) + 3(x_y)^2)$. Since $\|n_1(t)\|_{L^1} + \|n_2\|_{L^1} \lesssim t^{-1/2}$,

$$(13.25) \quad \begin{aligned} & \|\tilde{\mathcal{N}}_\lambda - \lambda^2(n_1, n_2)(\lambda^2 t, \lambda \cdot)\|_{H^{-1}} \\ &= \lambda \|\langle \eta \rangle^{-1} (\mathcal{F}_y n_1, \mathcal{F}_y n_2)(\lambda^2 t, \lambda^{-1} \eta)\|_{L^2(|\eta| \geq \lambda \eta_0)} \\ &\lesssim \lambda^{1/2} (\|n_1(\lambda^2 t, \cdot)\|_{L^1} + \|n_2(\lambda^2 t, \cdot)\|_{L^1}) \lesssim (\lambda t)^{-1/2}. \end{aligned}$$

Combining (13.25) with (13.24), we have

$$(13.26) \quad \begin{aligned} & \left\| \lambda^2 (\tilde{P}_1 n_1)(\lambda^2 t, \lambda \cdot) - 12 \{ (e^{4\lambda t \partial_y} d_{+, \lambda})^2 - (e^{-4\lambda t \partial_y} d_{-, \lambda})^2 \} \right\|_{H^{-1}} \\ &\lesssim C(t) (\lambda^{-1/2} + e^{-4a\lambda^2 t}), \end{aligned}$$

where $C(t)$ is a monotone decreasing function of t . Claim D.6 implies

$$\left\| b - \tilde{c} - \frac{1}{8} \tilde{P}_1(b^2) \right\|_Y \lesssim \|b\|_{L^\infty}^2 \|b\|_{L^2} \lesssim \|\mathbf{d}\|_{L^2}^2 \|\partial_y \mathbf{d}\|_{L^2},$$

whence

$$\lambda^2 \left\| \left(b - \tilde{c} - \frac{1}{8} \tilde{P}_1(b^2) \right) (\lambda^2 t, \lambda y) \right\|_Y \lesssim \lambda^{-1} \|\mathbf{d}_\lambda\|_{L^2}^2 \|\partial_y \mathbf{d}_\lambda\|_{L^2} \lesssim \lambda^{-1} t^{-5/4}.$$

We can obtain $\lambda^2 \left\| (I - \tilde{P}_1) b^2(\lambda^2 t, \lambda y) \right\|_{H^{-1}} \lesssim (\lambda t)^{-1/2}$ in the same as (13.25).

Combining the above with (13.24) and (13.25), we have

$$(13.27) \quad \begin{aligned} & \left\| \lambda^2 (\tilde{P}_1 n_2)(\lambda^2 t, \lambda \cdot) - 2 \{ (e^{4\lambda t \partial_y} d_{+, \lambda})^2 - 4(e^{4\lambda t \partial_y} d_{+, \lambda})(e^{-4\lambda t \partial_y} d_{-, \lambda}) \right. \\ & \quad \left. + (e^{-4\lambda t \partial_y} d_{-, \lambda})^2 \} \right\|_{H^{-1}} \lesssim C'(t) (\lambda^{-1/2} + e^{-4a\lambda^2 t}), \end{aligned}$$

where $C'(t)$ is a monotone decreasing function of t . Since

$$(13.28) \quad \left| 4i \Pi_*(\lambda^{-1} \eta)^{-1} - \frac{1}{4} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \right| \lesssim \lambda^{-1} \eta \quad \text{for } \lambda \geq 1,$$

it follows from (13.26) and (13.27) that

$$(13.29) \quad \begin{aligned} & \left\| A_5(\lambda^2 t, \lambda^{-1} D_y) \tilde{\mathcal{N}}_\lambda - 2 \begin{pmatrix} 2d_{+, \lambda}^2 - 2d_{+, \lambda}(e^{-8\lambda t \partial_y} d_{-, \lambda}) - (e^{-8\lambda t \partial_y} d_{-, \lambda})^2 \\ (e^{8\lambda t \partial_y} d_{+, \lambda})^2 + 2(e^{8\lambda t \partial_y} d_{+, \lambda})d_{-, \lambda} - 2d_{-, \lambda}^2 \end{pmatrix} \right\|_{H^{-2}} \\ &\lesssim C(t) (\lambda^{-1/2} + e^{-4a\lambda^2 t}), \end{aligned}$$

where $C(t)$ is a monotone decreasing function of t .

Next, we will show that $e^{\pm 4\lambda t \partial_y} d_{\pm, \lambda}$ locally tends to 0 around $y = \pm 4\lambda t$. Let $\alpha > 0$ and $\zeta_{\pm}(y) = 1 \pm \tanh \alpha y$. Then we have $0 \leq \zeta_{\pm}(y) \leq 2$, $0 < \pm \zeta'_{\pm}(y) \leq \alpha$ and $0 \leq |\zeta''_{\pm}(y)/\zeta'_{\pm}(y)| \leq 2\alpha$ for $y \in \mathbb{R}$. By (13.7),

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \zeta_+(y - 8\lambda t) d_{+, \lambda}^2(t, y) dy + 4\lambda \int_{\mathbb{R}} \zeta'_+(y - 8\lambda t) d_{+, \lambda}^2(t, y) dy \\ & \leq \int_{\mathbb{R}} \zeta''_+(y - 8\lambda t) (d_{+, \lambda})^2(t, y) dy - 2 \int_{\mathbb{R}} \zeta_+(y - 8\lambda t) (\partial_y d_{+, \lambda})^2(t, y) dy + V, \end{aligned}$$

where

$$\begin{aligned} V = & (2 + \alpha)\lambda \|A_3(\lambda^2 t, \lambda^{-1} D_y) \mathbf{d}_{\lambda}(t)\|_{L^2} \|\mathbf{d}_{\lambda}(t)\|_{H^1} \\ & + (2 + \alpha)\lambda \|A_4(\lambda^2 t, \lambda^{-1} D_y) \mathbf{d}_{\lambda}(t)\|_{L^2} \|\mathbf{d}_{\lambda}(t)\|_{H^1} \\ & + (2 + \alpha) \|\mathbf{d}_{\lambda}(t)\|_{H^1} \|A_5(\lambda^2 t, \lambda^{-1} D_y) (\tilde{\mathcal{N}}_{\lambda}(t) + \mathcal{N}_{\lambda}''(t))\|_{L^2} \\ & + 2 \|\mathbf{d}_{\lambda}(t)\|_{L^2} \|A_5(\lambda^2 t, \lambda^{-1} D_y) \mathcal{N}'_{\lambda}(t)\|_{L^2}. \end{aligned}$$

Using Lemma 13.1 and (13.9), we have

$$(13.30) \quad \lambda \|A_3(\lambda^2 t, \lambda^{-1} D_y) \mathbf{d}_{\lambda}(t)\|_{L^2} \lesssim \lambda^{-1/2} t^{-1}.$$

By Lemma 13.1, (13.11), (13.13)–(13.15) and (13.30),

$$V \lesssim \lambda^{-1/4} (t^{-9/8} + t^{-9/4}) + t^{-1} + t^{-3/2}.$$

If α is sufficiently small, it follows that

$$(13.31) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \zeta_+(y - 8\lambda t) d_{+, \lambda}^2(t, y) dy + 4\lambda \int_{\mathbb{R}} \zeta'_+(y - 8\lambda t) d_{+, \lambda}^2(t, y) dy \\ & \leq C \lambda^{-1/4} (t^{-9/8} + t^{-9/4}) + C(t^{-1} + t^{-3/2}), \end{aligned}$$

where C is a positive constant independent of $t > 0$ and $\lambda \geq 1$. Let $0 < t_1 < t_2 < \infty$. Integrating (13.31) over $[t_1, t_2]$, we obtain

$$(13.32) \quad 0 < \int_{t_1}^{t_2} \int_{\mathbb{R}} \zeta'_+(y - 8\lambda t) d_{+, \lambda}^2(t, y) dy dt \leq C(t_1, t_2) \lambda^{-1},$$

where $C(t_1, t_2)$ is a constant independent of $\lambda \geq 1$. We can prove

$$(13.33) \quad 0 < - \int_{t_1}^{t_2} \int_{\mathbb{R}} \zeta'_-(y + 8\lambda t) d_{-, \lambda}^2(t, y) dy dt \leq C(t_1, t_2) \lambda^{-1},$$

in exactly the same way. By (13.32) and (13.33),

$$\lim_{\lambda \rightarrow \infty} \|d_{\pm, \lambda}\|_{L^2([t_1, t_2] \times B_R^{\pm})} = 0$$

for any $R > 0$, where $B_R^{\pm} = \{y \in \mathbb{R} \mid |y \mp 8\lambda t| \leq R\}$. Combining the above with Corollary 13.2, (13.7), (13.21), (13.22) and (13.29), we see that \mathbf{d}_{∞} satisfies (13.16). \square

Now we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. By Corollary 13.2,

$$\|\partial_y(d_{\pm}^2)(t, \cdot)\|_{H^{-2}} \lesssim \|\mathbf{d}_{\infty}(t, \cdot)\|_{L^2}^2 \lesssim t^{-1/2},$$

whence $\partial_y(d_\pm(t)^2) \in L^1_{loc}((0, \infty); H^{-2}(\mathbb{R}))$. Combining the above with (13.16) and (13.17), we have $d_\pm(t) \in C([0, \infty); H^{-2}(\mathbb{R}))$ and

$$(13.34) \quad d_\pm(t) = c_\pm H_{2t} \pm 4 \int_0^t e^{2(t-s)\partial_y^2} \partial_y(d_\pm(s)^2) ds,$$

where $(c_+, c_-) = \sqrt{2\pi}(\mathcal{F}_y \tilde{\mathbf{d}}_1)(0, 0)$. If we choose $m_\pm \in (-2\sqrt{2}, 2\sqrt{2})$ so that

$$\int_{\mathbb{R}} u_B^\pm(t, y) dy = \frac{1}{2} \log \left(\frac{2\sqrt{2} \pm m_\pm}{2\sqrt{2} \mp m_\pm} \right) = c_\pm,$$

then $u_B^\pm(t)$ is also a solution to (13.34) satisfying $\sup_{t>0} t^{1/4} \|u_B^\pm(t)\|_{L^2} \lesssim \varepsilon$. Let $\|u\|_W = \sup_{t>0} t^{1/4} \|u(t, \cdot)\|_{L^2(\mathbb{R})}$. Since $\|\partial_y e^{2t\partial_y^2}\|_{B(L^1; L^2)} \lesssim t^{-3/4}$,

$$\begin{aligned} \|d_\pm - u_B^\pm\|_W &\leq 4 \sup_{t>0} t^{1/4} \int_0^t \|\partial_y e^{2(t-s)\partial_y^2}\|_{B(L^1; L^2)} \|d_\pm(s)^2 - u_B^\pm(s)^2\|_{L^1} ds \\ &\lesssim (\|d_\pm\|_W + \|u_B^\pm\|_W) \|d_\pm - u_B^\pm\|_W t^{1/4} \int_0^t (t-s)^{-3/4} s^{-1/2} ds \\ &\lesssim \varepsilon \|d_\pm - u_B^\pm\|_W. \end{aligned}$$

Thus we have $d_\pm(t, y) = u_B^\pm(t, y)$ for small ε and (13.3) follows from the uniqueness of the limiting profile $\mathbf{d}_\infty(t, y) = (d_+(t, y), d_-(t, y))$. Obviously $\mathbf{d}_\infty(t, y) = (u_B^+(t, y), u_B^-(t, y))$ satisfies (13.2). Now Theorem 1.3 follows immediately from (13.4), (13.24) and the definition of $b(t, y)$. Thus we complete the proof. \square

APPENDIX A. PROOF OF LEMMA 6.1

To prove Lemma 6.1, we need the following.

Claim A.1. *Let $\varphi_c(x) = c \operatorname{sech}^2(\sqrt{c/2}x)$, $\varphi = \varphi_2$ and $\partial_c^k \varphi = \partial_c^k \varphi_c|_{c=2}$ for $k \in \mathbb{N}$. Then*

$$(A.1) \quad \int_{\mathbb{R}} \varphi(x) dx = 4, \quad \int_{\mathbb{R}} \varphi(x)^2 dx = \frac{16}{3},$$

$$(A.2) \quad \int_{\mathbb{R}} \varphi(x) \partial_c \varphi(x) dx = - \int_{\mathbb{R}} \varphi'(x) \left(\int_{-\infty}^x \partial_c \varphi(z) dz \right) = 2,$$

$$(A.3) \quad \int_{\mathbb{R}} \varphi(x) \left(\int_x^\infty \partial_c \varphi(z) dz \right) dx = \int_{\mathbb{R}} \varphi(x) \left(\int_{-\infty}^x \partial_c \varphi(z) dz \right) = 2,$$

$$(A.4) \quad \int_{\mathbb{R}} \varphi(x) \left(\int_x^\infty \partial_c^2 \varphi(z) dz \right) dx = -\frac{1}{2}, \quad \int_{\mathbb{R}} \partial_c \varphi(x) \left(\int_{-\infty}^x \partial_c \varphi(z) dz \right) dx = \frac{1}{2},$$

$$(A.5) \quad \int_{\mathbb{R}} \left(\int_x^\infty \partial_c \varphi(z) dz \right) \left(\int_{-\infty}^x \partial_c \varphi(z) dz \right) dx = \frac{1}{6} - \frac{\pi^2}{36},$$

$$(A.6) \quad \int_{\mathbb{R}} \left(\int_x^\infty \partial_c^2 \varphi(z) dz \right) \left(\int_{-\infty}^x \partial_c \varphi(z) dz \right) dx = \frac{\pi^2}{96} - \frac{1}{16}.$$

Proof. Eq. (A.1) can be obtained by using the change of variable $s = \tanh x$. Since

$$(A.7) \quad \varphi_c(x) = \frac{c}{2} \varphi(\sqrt{c/2}x),$$

$$\int \varphi \partial_c \varphi dx = \frac{1}{2} \frac{d}{dc} \left(\frac{c}{2} \right)^{3/2} \Big|_{c=2} \int \varphi^2 dx = 2.$$

Using (A.7), we have

$$\begin{aligned}
 \partial_c \varphi(x) &= \frac{x}{4} \varphi'(x) + \frac{1}{2} \varphi(x) = \frac{1}{4} (x\varphi)' + \frac{1}{4} \varphi, \\
 (A.8) \quad \partial_c^2 \varphi(x) &= \frac{x^2}{16} \varphi''(x) + \frac{3x}{16} \varphi'(x) = \frac{1}{16} (x^2 \varphi' + x\varphi)' - \frac{1}{16} \varphi, \\
 \int_{\pm\infty}^x \partial_c \varphi &= \frac{x\varphi}{4} + \frac{\tanh x \mp 1}{2}, \quad \int_x^\infty \partial_c^2 \varphi = -\frac{x^2 \varphi' + x\varphi}{16} + \frac{\tanh x - 1}{8}.
 \end{aligned}$$

By (A.8) and the fact that φ is even,

$$\begin{aligned}
 \int \varphi \left(\int_x^\infty \partial_c \varphi \right) &= \int \varphi \left(\int_{-\infty}^x \partial_c \varphi \right) = \frac{1}{2} \int \varphi = 2, \\
 \int \varphi \int_x^\infty \partial_c^2 \varphi &= -\frac{1}{8} \int \varphi = -\frac{1}{2}, \\
 \int_{\mathbb{R}} \partial_c \varphi \int_{-\infty}^x \partial_c \varphi &= \frac{1}{8} \int \varphi = \frac{1}{2}.
 \end{aligned}$$

By (A.8),

$$\begin{aligned}
 &\int_{\mathbb{R}} \left(\int_x^\infty \partial_c \varphi(z) dz \right) \left(\int_{-\infty}^x \partial_c \varphi(z) dz \right) dx \\
 &= \int_{\mathbb{R}} \left\{ \frac{1}{4} - \left(\frac{x}{4} \varphi + \frac{1}{2} \tanh x \right)^2 \right\} dx \\
 &= \frac{1}{4} \int \operatorname{sech}^2 x - \frac{1}{2} \int x \operatorname{sech}^3 x \sinh x - \frac{1}{4} \int x^2 \operatorname{sech}^4 x dx \\
 &= -\frac{1}{4} \int x^2 \operatorname{sech}^4 x dx = -\frac{\pi^2 - 6}{36}.
 \end{aligned}$$

Here use the fact that $\int_0^\infty \frac{x}{e^x + 1} dx = \frac{\pi^2}{12}$. We have (A.6) in the same way. \square

Now we are in position to prove Lemma 6.1.

Proof of Lemma 6.1. By Claims A.1 and 2.1,

$$\begin{aligned}
 G_1 &= \int_{\mathbb{R}} \ell_1 \varphi_c(z) dz \\
 &= 3x_{yy} \int \varphi_c^2 - (c_t - 6c_y x_y) \int \varphi_c \partial_c \varphi_c \\
 &\quad + 3c_{yy} \int \varphi_c \int_z^\infty \partial_c \varphi_c + 3(c_y)^2 \int \varphi_c \int_z^\infty \partial_c^2 \varphi_c \\
 &= 16x_{yy} \left(\frac{c}{2} \right)^{3/2} - 2(c_t - 6c_y x_y) \left(\frac{c}{2} \right)^{1/2} + 6c_{yy} - \frac{3}{2} (c_y)^2 \left(\frac{2}{c} \right),
 \end{aligned}$$

and

$$\begin{aligned}
\left(\frac{c}{2}\right)^{-3/2} G_2 &= \int_{\mathbb{R}} \ell_1 \left(\int_{-\infty}^x \partial_c \varphi_c(z) dz \right) dz \\
&= (x_t - 2c - 3(x_y)^2) \int \varphi'_c \left(\int_{-\infty}^x \partial_c \varphi_c \right) \\
&\quad + 3x_{yy} \int \varphi_c \left(\int_{-\infty}^x \partial_c \varphi_c \right) - (c_t - 6c_y x_y) \int \partial_c \varphi_c \left(\int_{-\infty}^x \partial_c \varphi_c \right) \\
&\quad + 3c_{yy} \int \left(\int_z^\infty \partial_c \varphi_c \right) \left(\int_{-\infty}^x \partial_c \varphi_c \right) + 3(c_y)^2 \int \left(\int_z^\infty \partial_c^2 \varphi_c \right) \left(\int_{-\infty}^x \partial_c \varphi_c \right) \\
&= -2(x_t - 2c - 3(x_y)^2) \left(\frac{c}{2}\right)^{1/2} + 6x_{yy} - \frac{1}{2}(c_t - 6c_y x_y) \left(\frac{c}{2}\right)^{-1} + \mu_1 c_{yy} \left(\frac{c}{2}\right)^{-3/2} \\
&\quad + \mu_2 (c_y)^2 \left(\frac{c}{2}\right)^{-5/2}.
\end{aligned}$$

□

APPENDIX B. OPERATOR NORMS OF S_k^j AND $\tilde{\mathcal{C}}_k$

Claim B.1. *There exist positive constants η_1 and C such that for $\eta \in (0, \eta_1]$, $j \in \mathbb{Z}_{\geq 0}$, $k = 1, 2$ and $f \in L^2(\mathbb{R})$,*

$$(B.1) \quad \|\partial_y^j S_k^1[q_c](f)(t, \cdot)\|_Y \leq C \|e^{az} q_2\|_{L^2} \|\partial_y^j \tilde{P}_1 f\|_Y,$$

$$(B.2) \quad \|\partial_y^j S_k^1[q_c](f)(t, \cdot)\|_{Y_1} \leq C \|e^{az} q_2\|_{L^2} \|\partial_y^j \tilde{P}_1 f\|_{Y_1},$$

$$(B.3) \quad [\partial_y, S_k^1[q_c]] = 0.$$

Proof. Since the Fourier transform of $S_k^1 f$ can be written as

$$(B.4) \quad \mathcal{F}_y(S_k^1 f)(t, \eta) = \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) \hat{f}(\eta) \int dz q_2(z) \overline{g_{k1}^*(z, \eta, 2)},$$

we have $[\partial_y, S_k^1] = i\mathcal{F}_\eta^{-1}[\eta, \mathcal{F}_y(S_k^1 f)(t, \eta)] = 0$. Since

$$\sup_{\eta \in [-\eta_0, \eta_0]} \left| \int dz q_2(z) \overline{g_{k1}^*(z, \eta, 2)} \right| \lesssim \|e^{az} q_2(z)\|_{L^2}$$

by Claim 2.1, we see that (B.1) and (B.2) follow immediately from (B.4) and (B.3). □

Claim B.2. *There exist positive constants η_1 , δ and C such that if $\eta \in (0, \eta_1]$ and $\mathbb{M}_1(T) \leq \delta$, then for $k = 1, 2$, $t \in [0, T]$ and $f \in L^2(\mathbb{R})$,*

$$(B.5) \quad \|S_k^2[q_c](f)(t, \cdot)\|_{Y_1} \leq C \sup_{c \in [2-\delta, 2+\delta]} (\|e^{az} q_c\|_{L^2} + \|e^{az} \partial_c q_c\|_{L^2}) \|\tilde{c}\|_Y \|f\|_{L^2},$$

$$(B.6) \quad \begin{aligned} &\|\partial_y S_k^2[q_c](f)(t, \cdot)\|_{Y_1} \\ &\leq C \sum_{i=1,2} \sup_{c \in [2-\delta, 2+\delta]} \|e^{az} \partial_c^i q_c\|_{L^2} (\|c_y\|_Y \|f\|_{L^2} + \|\tilde{c}\|_Y \|\partial_y f\|_{L^2}), \end{aligned}$$

$$(B.7) \quad \|S_k^2[q_c](f)(t, \cdot)\|_Y \leq C \sum_{0 \leq i \leq 2} \sup_{c \in [2-\delta, 2+\delta]} \|e^{az} \partial_c^i q_c\|_{L^2} \|\tilde{c}\|_{L^\infty} \|f\|_{L^2},$$

$$(B.8) \quad \|[\partial_y, S_k^2[q_c]]f(t, \cdot)\|_{Y_1} \leq C \sum_{0 \leq i \leq 3} \sup_{c \in [2-\delta, 2+\delta]} \|e^{az} \partial_c^i q_c\|_{L^2} \|c_y\|_Y \|f\|_{L^2}.$$

Proof. By the definition of g_{k2}^* ,

$$\sup_{c \in [2-\delta, 2+\delta]} \sup_{|\eta| \leq \eta_0} \left| \int_{\mathbb{R}} g_{k2}^*(z, \eta, c) dz \right| \lesssim \sup_{c \in [2-\delta, 2+\delta]} (\|e^{az} q_c\|_{L_z^2} + \|e^{az} \partial_c q_c\|_{L_z^2}).$$

Since

$$\mathcal{F}_y(S_k^2[q_c]f)(t, \eta) = \int dy e^{-iy\eta} f(y) \tilde{c}(t, y) \frac{\mathbf{1}_{[-\eta_0, \eta_0]}(\eta)}{\sqrt{2\pi}} \int dz \overline{g_{k2}^*(z, \eta, c(t, y))},$$

we have

$$\begin{aligned} \|S_k^2[q_c](f)(t, \cdot)\|_{Y_1} &= \|\mathcal{F}_y(S_k^2[q_c](f))(t, \eta)\|_{L^\infty[-\eta_0, \eta_0]} \\ &\lesssim \sup_{c \in [2-\delta, 2+\delta]} (\|e^{az} q_c\|_{L_z^2} + \|e^{az} \partial_c q_c\|_{L_z^2}) \int |f(y) \tilde{c}(t, y)| dy \\ &\lesssim \sup_{c \in [2-\delta, 2+\delta]} (\|e^{az} q_c\|_{L_z^2} + \|e^{az} \partial_c q_c\|_{L_z^2}) \|f\|_{L^2} \|\tilde{c}\|_Y. \end{aligned}$$

Next, we will prove (B.7). Let

$$\begin{aligned} S_{k1}^2[q_c](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \tilde{c}(t, y_1) \overline{g_{k2}^*(z, \eta, 2)} e^{i(y-y_1)\eta} dy_1 dz d\eta, \\ S_{k2}^2[q_c](f)(t, y) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) \tilde{c}(t, y_1)^2 \overline{g_{k4}^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta, \end{aligned}$$

where

$$g_{k4}^*(z, \eta, c) = \frac{g_{k2}^*(z, \eta, c) - g_{k2}^*(z, \eta, 2)}{c - 2}.$$

Then $S_k^2[q_c] = S_{k1}^2[q_c] + S_{k2}^2[q_c]$ and we can prove

$$\begin{aligned} (B.9) \quad \|S_{k1}^2[q_c]f(t, \cdot)\|_Y &\lesssim \sum_{0 \leq i \leq 2} \sup_{c \in [2-\delta, 2+\delta]} \|e^{az} \partial_c^i q_c\|_{L_z^2} \|\tilde{c}\|_{L^\infty} \|f\|_{L^2}, \\ \|S_{k2}^2[q_c]f(t, \cdot)\|_{Y_1} &\lesssim \sum_{0 \leq i \leq 2} \sup_{c \in [2-\delta, 2+\delta]} \|e^{az} \partial_c^i q_c\|_{L_z^2} \|\tilde{c}\|_{L^4}^2 \|f\|_{L^2}, \end{aligned}$$

in exactly the same way as (B.1) and (B.5). Since

$$\|S_k^2[q_c]f(t, \cdot)\|_Y \lesssim \|S_{k1}^2[q_c]f(t, \cdot)\|_Y + \|S_{k2}^2[q_c]f(t, \cdot)\|_{Y_1},$$

(B.7) follows from (B.9).

Now we will show (B.8). Noting that

$$\begin{aligned} \mathcal{F}_y([\partial_y, S_{k1}^2[q_c]]f)(t, \eta) &= \frac{\mathbf{1}_{[-\eta_0, \eta_0]}(\eta)}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) c_y(t, y) e^{-iy\eta} dy \int_{\mathbb{R}} \overline{g_{k2}^*(z, \eta, 2)} dz, \\ \mathcal{F}_y([\partial_y, S_{k2}^2[q_c]]f)(t, \eta) &= \frac{\mathbf{1}_{[-\eta_0, \eta_0]}(\eta)}{\sqrt{2\pi}} \int_{\mathbb{R}^2} f(y) \partial_y \left(\tilde{c}(t, y)^2 \overline{g_{k4}^*(z, \eta, c(t, y))} \right) \\ &\quad \times e^{-iy\eta} dz dy, \end{aligned}$$

we can prove (B.8) in the same way as (B.5). Eq. (B.6) immediately follows from (B.5) and (B.8). Thus we complete the proof. \square

Next we will estimate the operator norm of $S^3[p](f)$.

Claim B.3. *There exist positive constants η_1 and C such that for $\eta_0 \in (0, \eta_1]$, $k = 1, 2$, $t \geq 0$ and $f \in L^2(\mathbb{R})$,*

$$(B.10) \quad \|S_k^3[p](f)(t, \cdot)\|_Y \leq C e^{-a(4t+L)} \|e^{az} p\|_{L^2} \|\tilde{P}_1 f\|_Y,$$

$$(B.11) \quad \|S_k^3[p](f)(t, \cdot)\|_{Y_1} \leq C e^{-a(4t+L)} \|e^{az} p\|_{L^2} \|\tilde{P}_1 f\|_{Y_1}.$$

Moreover,

$$(B.12) \quad [\partial_y, S_{k1}^3[p]] = 0.$$

Proof. The Fourier transform of $S_k^3 f$ is

$$(B.13) \quad \mathcal{F}_y(S_k^3 f)(t, \eta) = \mathbf{1}_{[-\eta_0, \eta_0]}(\eta) \hat{f}(\eta) \int_{\mathbb{R}} p(z + 4t + L) \overline{g_k^*(z, \eta)} dz.$$

By Claim 2.1,

$$(B.14) \quad \left| \int p(z + 4t + L) \overline{g_k^*(z, \eta)} dz \right| \leq e^{-a(4t+L)} \|e^{az} p(z)\|_{L_z^2} \sup_{|\eta| \leq \eta_0} \|e^{-az} g_k^*(z, \eta)\|_{L_z^2} \\ \lesssim e^{-a(4t+L)} \|e^{az} p(z)\|_{L^2}.$$

Combining (B.13) and (B.14), we immediately have (B.10) and (B.11). Eq. (B.12) clearly follows from the definition of S_k^3 . Thus we complete the proof. \square

Claim B.4. *There exist positive constants η_1 , δ and C such that if $\eta_0 \in (0, \eta_1]$ and $M_1(T) \leq \delta$, then for $k = 1, 2$, $t \in [0, T]$ and $f \in L^2$,*

$$(B.15) \quad \|S_k^4[p](f)(t, \cdot)\|_{Y_1} \leq C e^{-a(4t+L)} \|e^{az} p\|_{L^2} \|\tilde{c}\|_Y \|f\|_{L^2},$$

$$(B.16) \quad \|[\partial_y, S_{k2}^4[p]]f(t, \cdot)\|_{Y_1} \leq C e^{-a(4t+L)} \|e^{az} p\|_{L^2} \|c_y\|_Y \|f\|_{L^2}.$$

Proof. Since

$$\begin{aligned} & \mathcal{F}_y(S_k^4[p](f))(t, \eta) \\ &= \frac{\mathbf{1}_{[-\eta_0, \eta_0]}(\eta)}{\sqrt{2\pi}} \int_{\mathbb{R}^2} f(y) \tilde{c}(t, y) p(z + 4t + L) \overline{g_{k3}^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy, \end{aligned}$$

we have

$$\|S_{k2}^4[p](f)(t, \cdot)\|_{Y_1} \lesssim \|f\|_{L^2} \|\tilde{c}\|_{L^2} e^{-a(4t+L)} \|e^{az} p\|_{L^2} \sup_{\substack{c \in [2-\delta, 2+\delta], \\ \eta \in [-\eta_0, \eta_0]}} \|e^{-az} g_{k3}^*(z, \eta, c)\|_{L_z^2} \\ \lesssim e^{-a(4t+L)} \|f\|_{L^2} \|\tilde{c}(t)\|_Y.$$

Noting that

$$\begin{aligned} [\partial_y, S_{k2}^4[p]](f) &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} f(y_1) p(z + 4t + L) \partial_{y_1} \{c(t, y_1) \overline{g_{k3}^*(z, \eta, c(t, y_1))}\} \\ &\quad \times e^{i(y-y_1)\eta} dy_1 dz d\eta, \end{aligned}$$

we can prove (B.16) in the same way as (B.15). Thus we complete the proof. \square

Claim B.5. *There exist positive constants η_1 , δ and C such that if $\eta_0 \in (0, \eta_1]$ and $M_1(T) \leq \delta$, then for $k = 1, 2$ and $t \in [0, T]$,*

$$\|S_k^5 f\|_{Y_1} + \|S_k^6 f\|_{Y_1} \leq C \|v(t, \cdot)\|_X \|f\|_{L^2}.$$

Proof. Using the Schwarz inequality, we have

$$\begin{aligned} \|S_k^6 f\|_{Y_1} &= \sup_{|\eta| \leq \eta_0} \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}^2} v(t, z, y) f(y) \overline{\partial_z g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \right| \\ &\lesssim \sup_{c \in [2-\delta, 2+\delta], \eta \in [-\eta_0, \eta_0]} \|e^{-az} \partial_z g_k^*(z, \eta, c)\|_{L_z^2} \|v(t)\|_X \|f\|_{L_y^2}. \end{aligned}$$

Since $\|e^{-az} \partial_z g_k^*(z, \eta, c)\|_{L_z^2}$ is bounded for $c \in (1, 3)$ and $\eta \in [-\eta_0, \eta_0]$, we have

$$\|S_k^6 f\|_{Y_1} \lesssim \|v(t, \cdot)\|_X \|f\|_{L^2}.$$

We can estimate S_k^5 in exactly the same way. Thus we complete the proof. \square

Next, we will estimate operator norms of $\tilde{\mathcal{C}}_k$ ($k = 1, 2$).

Claim B.6. *There exist positive constants δ and C such that if $\sup_{t \in [0, T]} \|\tilde{c}(t)\|_Y \leq \delta$, then for $k = 1, 2$ and $t \in [0, T]$,*

$$\begin{aligned} \|\mathcal{C}_k f\|_Y &\leq C \|\tilde{c}\|_{L^\infty} \|\tilde{P}_1 f\|_Y, \\ \|\mathcal{C}_k f\|_{Y_1} &\leq C \|\tilde{c}\|_Y \|\tilde{P}_1 f\|_Y. \end{aligned}$$

Proof. Since $\|\tilde{c}\|_{L^\infty} \lesssim \|\tilde{c}\|_Y$ by Remark 4.1, it follows that $|c^2 - 4| \leq (2 + O(\delta))|\tilde{c}|$. Thus we have

$$\|\mathcal{C}_1 f\|_Y \leq \frac{1}{2} \|c^2 - 4\|_{L^\infty} \|\tilde{P}_1 f\|_{L^2} \lesssim \|\tilde{c}\|_{L^\infty} \|\tilde{P}_1 f\|_Y,$$

$$\begin{aligned} \|\mathcal{C}_1 f\|_{Y_1} &= \frac{1}{2} \left\| \mathcal{F}(c^2 - 4) * \mathcal{F}(\tilde{P}_1 f) \right\|_{L^\infty([- \eta_0, \eta_0])} \\ &\lesssim \|c^2 - 4\|_{L^2} \|\tilde{P}_1 f\|_{L^2} \lesssim \|\tilde{c}\|_Y \|\tilde{P}_1 f\|_Y. \end{aligned}$$

We can estimate \mathcal{C}_2 in exactly the same way. Thus we complete the proof. \square

Claim B.7. *There exist positive constants δ and C such that if $\sup_{t \in [0, T]} \|\tilde{c}(t)\|_Y \leq \delta$, then for $k = 1, 2$ and $t \in [0, T]$,*

$$\begin{aligned} \|[\partial_y, \mathcal{C}_k] f\|_Y &\leq C \|c_y\|_{L^\infty} \|f\|_{L^2}, \\ \|[\partial_y, \mathcal{C}_k] f\|_{Y_1} &\leq C \|c_y\|_Y \|f\|_{L^2}. \end{aligned}$$

Proof. Since $[\partial_y, \mathcal{C}_1] = \tilde{P}_1 c c_y \tilde{P}_1$,

$$\|[\partial_y, \mathcal{C}_1] f\|_Y \lesssim \|c_y\|_{L^\infty} \|f\|_{L^2},$$

$$\|[\partial_y, \mathcal{C}_1] f\|_{Y_1} = \left\| \mathcal{F}(c c_y) * \mathcal{F}(\tilde{P}_1 f) \right\|_{L^\infty([- \eta_0, \eta_0])} \lesssim \|c_y\|_Y \|\tilde{P}_1 f\|_Y.$$

We can prove the estimate for $[\partial_y, \mathcal{C}_2]$ in the same way. Thus we complete the proof. \square

APPENDIX C. PROOF OF CLAIMS 7.1, 6.2 AND 6.3

Proof of Claims 6.2 and 6.3. Claims B.1–B.3 and 6.1 imply that for $s \in [0, T]$,

$$(C.1) \quad \begin{aligned} \|\bar{S}_1\|_{B(Y)} &\lesssim \|\tilde{S}_1\|_{B(Y)} \|(1 + \tilde{\mathcal{C}}_2)^{-1}\|_{B(Y)} \\ &\lesssim \sum_{k=1,2} (\|S_k^1[\partial_c \varphi_c]\|_{B(Y)} + \|S_k^1[\varphi'_c]\|_{B(Y)}) \lesssim 1, \end{aligned}$$

$$(C.2) \quad \begin{aligned} \|\bar{S}_2\|_{B(Y, Y_1)} &\lesssim \|\tilde{S}_2\|_{B(Y, Y_1)} \|(1 + \tilde{\mathcal{C}}_2)^{-1}\|_{B(Y)} \\ &\lesssim \sum_{k=1,2} (\|S_k^2[\partial_c \varphi_c]\|_{B(Y)} + \|S_k^2[\varphi'_c]\|_{B(Y)}) \\ &\lesssim \|\tilde{c}\|_Y \lesssim \mathbb{M}_1(T) \langle s \rangle^{-1/4}, \end{aligned}$$

$$(C.3) \quad \begin{aligned} \|\bar{S}_3\|_{B(Y)} &\lesssim \sum_{k=1,2} \|S_k^3[\psi]\|_{B(Y)} \lesssim e^{-a(4s+L)}, \\ \|\bar{S}_3\|_{B(Y_1)} &\lesssim \sum_{k=1,2} \|S_k^3[\psi]\|_{B(Y_1)} \lesssim e^{-a(4s+L)}. \end{aligned}$$

By Claims B.4 and 6.1,

$$\begin{aligned} \|\bar{S}_4\|_{B(Y, Y_1)} &\lesssim \sum_{k=1,2} \left(\|S_k^3[\psi]((\sqrt{2/c} - 1) \cdot)\|_{B(Y, Y_1)} + \|S_k^4[\psi](c^{-1/2} \cdot)\|_{B(Y, Y_1)} \right) \\ &\quad + \sum_{k=1,2} \left\| (S_k^3[\psi'] + S_k^4[\psi'])((\sqrt{c} - \sqrt{2}) \cdot) \right\|_{B(Y, Y_1)} \\ &\lesssim \sum_{k=1,2} (\|S_k^3[\psi]\|_{B(Y_1)} + \|S_k^3[\psi']\|_{B(Y_1)}) \|\tilde{c}\|_Y \\ &\quad + \sum_{k=1,2} (\|S_k^4[\psi]\|_{B(Y, Y_1)} + \|S_k^4[\psi']\|_{B(Y, Y_1)}) \|\tilde{c}\|_{L^\infty}. \end{aligned}$$

Thus we have

$$(C.4) \quad \|\bar{S}_4\|_{B(Y, Y_1)} \lesssim \mathbb{M}_1(T) \langle s \rangle^{-1/4} e^{-a(4s+R)}.$$

By Claims B.5 and 6.1,

$$(C.5) \quad \|\bar{S}_5\|_{B(Y, Y_1)} \lesssim \sum_{k=1,2} (\|S_k^5\|_{B(Y, Y_1)} + \|S_k^6\|_{B(Y, Y_1)}) \lesssim \mathbb{M}_2(T) \langle s \rangle^{-3/4}.$$

Obviously,

$$(C.6) \quad \|\partial_y\|_{B(Y)} + \|\partial_y\|_{B(Y_1)} \lesssim \eta_0.$$

By (6.15), (6.17), (C.1)–(C.6) and the fact that $Y_1 \subset Y$,

$$\begin{aligned} \|B_3 - B_1\|_{B(Y)} &\leq \|\tilde{\mathcal{C}}_1\|_{B(Y)} + \eta_0^2 \sum_{j=1,2} \|\bar{S}_j\|_{B(Y)} + \sum_{j=3,4,5} \|\bar{S}_j\|_{B(Y)} \\ &\lesssim \mathbb{M}_1(T) + \mathbb{M}_2(T) + \eta_0^2 + e^{-aL}. \end{aligned}$$

Since B_1 is invertible, we see that $\|B_3^{-1}\|_{B(Y)}$ is bounded for $t \in [0, T]$ if $\|B_3 - B_1\|_{B(Y)}$ remains small on $[0, T]$. We can prove the boundedness of $\|B_3^{-1}\|_{B(Y_1)}$ in the same way. This completes the proof of Claim 6.2.

Using Claims B.1 and B.3, we can prove

$$(C.7) \quad \|\tilde{S}_1\|_{B(Y)} + \|\tilde{S}_1\|_{B(Y_1)} \lesssim 1,$$

$$(C.8) \quad \|\tilde{S}_3\|_{B(Y)} + \|\tilde{S}_3\|_{B(Y_1)} \lesssim e^{-a(4t+L)} \quad \text{for } t \geq 0,$$

in the same way as (C.1) and (C.3). Claim 6.3 immediately follows from (C.6), (C.7) and (C.8). Thus we complete the proof. \square

Proof of Claim 7.1. In view of (6.15),

$$[\partial_y, B_3] = [\partial_y, \tilde{C}_1] + \sum_{j=1,2} \partial_y^2 [\partial_y, \tilde{S}_j] - \sum_{j=3,4,5} [\partial_y, \tilde{S}_j],$$

Now we will estimate each term of the right hand side. By Claim B.7 and the definition of \tilde{C}_k ,

$$(C.9) \quad \|[\partial_y, \tilde{C}_k]\|_{B(Y, Y_1)} \lesssim \mathbb{M}_1(T) \langle s \rangle^{-3/4} \quad \text{for } k = 1, 2 \text{ and } s \in [0, T].$$

Since $[\partial_y, \tilde{S}_1] = 0$ by (B.3), we have $[\partial_y, \tilde{S}_1] = \tilde{S}_1[\tilde{C}_2, \partial_y](1 + \tilde{C}_2)^{-1}$. Thus by Claim 6.1, (C.1) and (C.9),

$$(C.10) \quad \begin{aligned} \|[\partial_y, \tilde{S}_1]\|_{B(Y, Y_1)} &\lesssim \|\tilde{S}_1\|_{B(Y_1)} \|\tilde{C}_2, \partial_y\|_{B(Y, Y_1)} \|(1 + \tilde{C}_2)^{-1}\|_{B(Y)} \\ &\lesssim \mathbb{M}_1(T) \langle s \rangle^{-3/4} \quad \text{for } s \in [0, T]. \end{aligned}$$

Applying Claims 6.1, B.2, B.7 and (C.2) to $[\partial_y, \tilde{S}_2] = \{[\partial_y, \tilde{S}_2] + \tilde{S}_2[\tilde{C}_2, \partial_y]\}(I + \tilde{C}_2)^{-1}$, we obtain

$$(C.11) \quad \begin{aligned} \|[\partial_y, \tilde{S}_2]\|_{B(Y, Y_1)} &\lesssim \|[\partial_y, \tilde{S}_2]\|_{B(Y, Y_1)} + \|\tilde{S}_2\|_{B(Y, Y_1)} \|[\partial_y, \tilde{C}_2]\|_{B(Y)} \\ &\lesssim \sum_{k=1,2} \left(\|[\partial_y, S_k^2[\partial_c \varphi_c]]\|_{B(Y, Y_1)} + \|[\partial_y, S_k^2[\varphi'_c]]\|_{B(Y, Y_1)} \right) \\ &\quad + \|\tilde{S}_2\|_{B(Y, Y_1)} \|[\partial_y, C_2]\|_{B(Y)} \\ &\lesssim \|c_y\|_Y + \|\tilde{c}\|_Y \|c_y\|_{L^\infty} \lesssim \mathbb{M}_1(T) \langle s \rangle^{-3/4} \quad \text{for } s \in [0, T]. \end{aligned}$$

Since $[\partial_y, S_k^3] = 0$ by (B.12), we have $[\partial_y, \tilde{S}_3] = \tilde{S}_3[\tilde{C}_2, \partial_y](I + \tilde{C}_2)^{-1}$. Hence it follows from Claims 6.1, B.3, (C.3) and (C.9) that

$$(C.12) \quad \|[\partial_y, \tilde{S}_3]\|_{B(Y, Y_1)} \lesssim \|\tilde{S}_3\|_{B(Y_1)} \|c_y\|_Y \lesssim \mathbb{M}_1(T) \langle s \rangle^{-3/4} e^{-a(4s+L)}.$$

By (C.4), (C.5) and (C.6), we have for $s \in [0, T]$,

$$(C.13) \quad \|[\partial_y, \tilde{S}_4]\|_{B(Y, Y_1)} \lesssim \eta_0 \mathbb{M}_1(T) \langle s \rangle^{-1/4} e^{-a(4s+L)},$$

$$(C.14) \quad \|[\partial_y, \tilde{S}_5]\|_{B(Y, Y_1)} \lesssim \eta_0 \mathbb{M}_2(T) \langle s \rangle^{-3/4}.$$

Combining (C.9)–(C.14), we obtain Claim 7.1. Thus we complete the proof. \square

APPENDIX D. ESTIMATES OF R^k

Claim D.1. *There exist positive constants δ and C such that if $\mathbb{M}_1(T) \leq \delta$, then for $t \in [0, T]$,*

$$\|R_k^2(t, \cdot)\|_{Y_1} \leq CM_1(T)^2 \langle t \rangle^{-1}, \quad \|\partial_y R_k^2(t, \cdot)\|_{Y_1} \leq CM_1(T)^2 \langle t \rangle^{-5/4}.$$

Proof. By Claims B.1, B.2 and (4.2),

$$\begin{aligned} \|R_k^2\|_{Y_1} &\lesssim \|\tilde{c}\|_Y (\|x_{yy}\|_Y + \|c_{yy}\|_Y) + \|c_y\|_Y^2 (1 + \|\tilde{c}\|_{L^\infty}) \\ &\lesssim \mathbb{M}_1(T)^2 \langle s \rangle^{-1}. \end{aligned}$$

We can estimate $\|\partial_y R_k^2\|_{Y_1}$ in the same way. Thus we complete the proof. \square

Claim D.2. *There exist positive constants δ and C such that if $\mathbb{M}_1(T) \leq \delta$, then for $t \in [0, T]$, $\|R_k^3(t, \cdot)\|_{Y_1} \leq C \langle t \rangle^{-1/2} e^{-a(4t+L)} \mathbb{M}_1(T)^2$.*

Proof. We decompose R_k^3 into three terms. Let

$$\begin{aligned} R_{k1}^3 &= \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \left\{ \partial_z^3 \tilde{\psi}_{c(t, y_1)}(z) - 3c_{yy}(t, y_1) \int_z^\infty \partial_c \tilde{\psi}_{c(t, y_1)}(z_1) dz_1 \right\} \\ &\quad \times \overline{g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta \\ &\quad - \frac{1}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \left\{ \tilde{c}(t, y_1) \psi'''(z + 4t + L) - 3c_{yy}(t, y_1) \int_{z+4t+L}^\infty \psi(z_1) dz_1 \right\} \\ &\quad \times \overline{g_k^*(z, \eta)} e^{i(y-y_1)\eta} dy_1 dz d\eta, \\ R_{k2}^3 &= -\frac{3}{2\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \left\{ \partial_z (\tilde{\psi}_{c(t, y_1)}^2) + x_{yy} \tilde{\psi}_{c(t, y_1)} + 3c_y(t, y_1)^2 \int_z^\infty \partial_c^2 \tilde{\psi}_{c(t, y_1)} \right\} \\ &\quad \times \overline{g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dy_1 dz d\eta, \end{aligned}$$

and

$$\begin{aligned} R_{k3}^3 &= -\frac{3}{\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \left(\sqrt{2c(t, y_1)} - 2 - \tilde{c}(t, y_1) \right) \psi(z + 4t + L) \varphi_{c(t, y_1)}(z) \\ &\quad \times \overline{\partial_z g_k^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy d\eta \\ &\quad - \frac{3}{\pi} \int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}^2} \tilde{c}(t, y_1)^2 \psi(z + 4t + L) \overline{g_{k5}^*(z, \eta, c(t, y_1))} e^{i(y-y_1)\eta} dz dy d\eta, \end{aligned}$$

where $g_{k5}(z, \eta, c)^* = \{\varphi_c(z) \partial_z g_k^*(z, \eta, c) - \varphi(z) \partial_z g_k^*(z, \eta)\} / \tilde{c}$. Then $R_k^3 = \sum_{i=1}^3 R_{ki}^3$.

Let us estimate R_{k1}^3 by using Claims B.3 and B.4. Since $\tilde{\psi}_c(z) = (2\sqrt{2c}-2)\psi(z+4t+L)$, we have

$$\begin{aligned} R_{k1}^3 &= S_k^3[\psi'''] \left(2\sqrt{2c} - 4 - \tilde{c} \right) + S_k^4[\psi'''] \left(2\sqrt{2c} - 4 \right) \\ &\quad + 3S_k^3[\partial_z^{-1}\psi] \left(((2/c)^{1/2} - 1)c_{yy} \right) + 3S_k^4[\partial_z^{-1}\psi] \left((2/c)^{1/2} c_{yy} \right), \\ R_{k2}^3 &= -24(S_k^3 + S_k^4)[(\psi^2)']((\sqrt{c} - \sqrt{2})^2) - 6\sqrt{2}(S_k^3 + S_k^4)[\psi]((\sqrt{c} - \sqrt{2})x_{yy}) \\ &\quad - \frac{3\sqrt{2}}{2}(S_k^3 + S_k^4)[\partial_z^{-1}\psi](c^{-3/2}(c_y)^2). \end{aligned}$$

Since $2\sqrt{2c} - 4 = \tilde{c} + O(\tilde{c}^2)$ and $(2/c)^{1/2} - 1 = O(\tilde{c})$ and $\tilde{P}_1 L^1 \subset Y_1$, it follows from Claim B.3 that

$$\begin{aligned} \|S_k^3[\psi'''] (2\sqrt{2c} - 4 - \tilde{c})\|_{Y_1} &\lesssim e^{-a(4t+L)} \|\tilde{c}\|_Y^2, \\ \left\| \sqrt{2} S_k^3[\partial_z^{-1}\psi] \left((2/c)^{1/2} - 1 \right) c_{yy} \right\|_{Y_1} &\lesssim e^{-a(4t+L)} \|\tilde{c}\|_Y \|c_{yy}\|_Y. \end{aligned}$$

By Claim B.4,

$$\|S_k^4[\psi'''](\sqrt{c} - \sqrt{2})\|_{Y_1} + \|S_k^4[\partial_z^{-1}\psi](c^{-1/2}c_{yy})\|_{Y_1} \lesssim e^{-a(4t+L)} \|\tilde{c}\|_Y (\|\tilde{c}\|_Y + \|c_{yy}\|_Y).$$

Thus we prove $\|R_{k1}^3\|_{Y_1} \lesssim e^{-a(4t+L)} \|\tilde{c}\|_Y (\|\tilde{c}\|_Y + \|c_{yy}\|_Y)$. Similarly, we have

$$\|R_{k2}^3\|_{Y_1} \lesssim e^{-a(4t+L)} (\|\tilde{c}\|_Y^2 + \|\tilde{c}\|_Y \|x_{yy}\|_Y + \|c_y\|_Y^2), \quad \|R_{k3}^3\|_{Y_1} \lesssim e^{-a(4t+L)} \|\tilde{c}\|_Y^2.$$

Thus we complete the proof. \square

Claim D.3. *There exists a positive constant C such that*

$$\|\tilde{\mathcal{A}}_1(t)\|_{B(Y)} + \|\tilde{\mathcal{A}}_1(t)\|_{B(Y_1)} \leq Ce^{-a(4t+L)} \quad \text{for every } t \geq 0 \text{ and } L \geq 0.$$

Proof. In view of (6.7),

$$\begin{aligned} \tilde{a}_k(t, D_y) \tilde{c} = & S_k^3[\psi'''](\tilde{c}) + 3S_k^3[\partial_z^{-1}\psi](c_{yy}) \\ & - 6\mathcal{F}_\eta^{-1} \left\{ \int \varphi(z) \psi(z + 4t + L) \overline{\partial_z g_k^*(z, \eta)} dz (\mathcal{F}_y \tilde{c})(t, \eta) \right\}. \end{aligned}$$

Hence it follows from Claim B.3 and (B.14) that

$$\|\tilde{a}_k(t, D_y)\|_{B(Y)} + \|\tilde{a}_k(t, D_y)\|_{B(Y_1)} \lesssim e^{-a(4t+L)}.$$

Thus we complete the proof of Claim D.3. \square

Claim D.4. *There exist positive constants C and L_0 such that if $L \geq L_0$, then*

$$\|A_1(t)\|_{B(Y)} \leq Ce^{-a(4t+L)} \quad \text{for every } t \geq 0.$$

Proof. Since B_1 is invertible and $\|\tilde{S}_3\|_{B(Y)} \lesssim \sum_{k=1,2} \|S_k^3[\psi]\|_{B(Y)} \lesssim e^{-a(4t+L)}$, we have Claim D.4. \square

Claim D.5. *Suppose $a \in (0, 1)$ and $\mathbb{M}_1(T) \leq \delta$. If δ is sufficiently small, then there exists a positive constant C such that*

$$(D.1) \quad \|R_k^4(t)\|_{Y_1} \leq C(\mathbb{M}_1(T) + \mathbb{M}_2(T))\mathbb{M}_2(T)\langle t \rangle^{-3/2},$$

$$(D.2) \quad \|R_k^5(t)\|_{Y_1} \leq CM_1(T)\mathbb{M}_2(T)\langle t \rangle^{-1},$$

$$(D.3) \quad \|R_k^6\|_{Y_1} \leq Ce^{-a(4t+L)}\langle t \rangle^{-1}\mathbb{M}_1(T)\mathbb{M}_2(T),$$

$$(D.4) \quad \|R_k^5(t)\|_Y \leq CM_1(T)\mathbb{M}_2(T)\langle t \rangle^{-5/4}, .$$

Proof. By Lemma 2.2 and (5.3), we can rewrite II_k^1 as $II_k^1 = i\eta II_{k1}^1 + II_{k2}^1 + II_{k3}^1$, where

$$II_{k1}^1(t, \eta) = -6 \int_{\mathbb{R}} c_y(t, y) h_{1k}(t, y, \eta) e^{-iy\eta} dy,$$

$$II_{k2}^1(t, \eta) = 3 \int_{\mathbb{R}} c_{yy}(t, y) h_{1k}(t, y, \eta) e^{-iy\eta} dy,$$

$$II_{k3}^1(t, \eta) = 3 \int_{\mathbb{R}} (c_y(t, y))^2 h_{2k}(t, y, \eta) e^{-iy\eta} dy,$$

$$h_{jk}(t, y, \eta) = \int_{\mathbb{R}} v(t, z, y) \left(\int_{-\infty}^z \overline{\partial_c^j g_k^*(z_1, \eta, c(t, y))} dz_1 \right) dz \quad \text{for } j = 1, 2.$$

First, we will estimate $II_k^1(t, \cdot)$. Since

$$\sup_{-\eta_0 \leq \eta \leq \eta_0, 2-\delta \leq c \leq 2+\delta} \left\| e^{-az} \int_{-\infty}^z g_k^*(z_1, \eta, c) dz_1 \right\|_{L_z^2} < \infty,$$

there exists a positive constant C such that

$$\sup_{\eta_0 \leq \eta \leq \eta_0} |h_{jk}(t, y, \eta)| \leq C \|e^{az} v(t, z, y)\|_{L_z^2} \quad \text{for any } y \in \mathbb{R} \text{ and } t \geq 0.$$

Thus by the Schwarz inequality,

$$(D.5) \quad \|II_{k1}^1(t, \cdot)\|_{L_\eta^\infty(-\eta_0, \eta_0)} \leq \|c_y(t)\|_Y \sup_{\eta \in [-\eta_0, \eta_0]} \left(\int_{\mathbb{R}} |h_{1k}(t, y, \eta)|^2 dy \right)^{1/2} \\ \lesssim \|c_y(t)\|_Y \|v(t)\|_X.$$

We can prove

$$(D.6) \quad \|II_{k2}^1(t, \eta)\|_{L_\eta^\infty[-\eta_0, \eta_0]} \lesssim \|c_{yy}\|_Y \|v(t, \cdot)\|_X,$$

$$(D.7) \quad \|II_{k3}^1(t, \eta)\|_{L_\eta^\infty[-\eta_0, \eta_0]} \lesssim \|c_y\|_{L^4(\mathbb{R})}^2 \|v(t, \cdot)\|_X,$$

in exactly the same way.

Next, we will estimate II_k^2 and II_k^3 . Since

$$(D.8) \quad \sup_{c \in [2-\delta, 2+\delta], \eta \in [-\eta_0, \eta_0]} \|e^{-2az} g_k^*(z, \eta, c)\|_{L_z^\infty} < \infty,$$

$$(D.9) \quad \sup_{c \in [2-\delta, 2+\delta], \eta \in [-\eta_0, \eta_0]} (\|e^{-az} g_k^*\|_{L^2} + \|e^{-az} \partial_z g_k^*\|_{L^2} + \|e^{-az} \partial_c g_k^*\|_{L^2}) < \infty,$$

we have

$$\|II_k^2\|_{L^\infty[-\eta_0, \eta_0]} = 3 \sup_{\eta \in [-\eta_0, \eta_0]} \left| \int_{\mathbb{R}^2} v(t, z, y)^2 \partial_z \overline{g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \right| \\ \lesssim \|v\|_X^2 \sup_{c \in [2-\delta, 2+\delta], \eta \in [-\eta_0, \eta_0]} \|e^{-2az} g_k^*(z, \eta, c)\|_{L_z^\infty} \\ \lesssim \|v\|_X^2.$$

and

$$\|II_{k1}^3\|_{L_\eta^\infty[-\eta_0, \eta_0]} \lesssim \|v\|_X \|x_{yy}\|_Y, \quad \|R_k^5\|_{Y_1} \lesssim \|II_{k2}^3\|_{L_\eta^\infty[-\eta_0, \eta_0]} \lesssim \|v\|_X \|x_y\|_Y.$$

Combining the above, we have

$$\|R_k^4(t)\|_{Y_1} \lesssim \sup_{-\eta_0 \leq \eta \leq \eta_0} (|II_k^1(t, \eta)| + |II_k^2(t, \eta)| + |II_{k1}^3(t, \eta)|) \\ \lesssim \|v(t, \cdot)\|_X (\|c_y(t)\|_Y + \|c_{yy}(t)\|_Y + \|c_y(t)\|_{L^4}^2 \\ + \|x_{yy}\|_Y) + \|v(t, \cdot)\|_X^2,$$

which implies (D.1).

By the Schwarz inequality and (8.11),

$$\|R_k^6\|_{Y_1} \lesssim \sup_{|\eta| \leq \eta_0} \left| \int_{\mathbb{R}^2} v(t, x, y) \tilde{\psi}_{c(t, y)} \overline{\partial_z g_k^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy \right| \\ \lesssim \|v(t)\|_X \|\tilde{\psi}_{c(t, y)}\|_X \sup_{c \in [2-\delta, 2+\delta], \eta \in [-\eta_0, \eta_0]} \|e^{-2az} g_k^*(z, \eta, c)\|_{L_z^\infty} \\ \lesssim e^{-a(4t+L)} \|\tilde{c}(t)\|_{L^2(\mathbb{R})} \|v(t)\|_X.$$

Finally, we will estimate $\|R_k^5\|_Y$. Let

$$II_{k21}^3 = 6 \int_{\mathbb{R}^2} v(t, z, y) x_y(t, y) \overline{g_k^*(z, \eta)} e^{-iy\eta} dz dy \\ = 6\sqrt{2\pi} \int_{\mathbb{R}} \mathcal{F}_y(x_y(t, \cdot) v(t, z, \cdot)) (\eta) \overline{g_k^*(z, \eta)} dz, \\ II_{k22}^3 = 6 \int_{\mathbb{R}^2} v(t, z, y) x_y(t, y) \tilde{c}(t, y) \overline{g_{k3}^*(z, \eta, c(t, y))} e^{-iy\eta} dz dy.$$

Then $II_{k2}^3 = II_{k21}^3 + II_{k22}^3$. By the Schwarz inequality,

$$|II_{k21}^3| \lesssim \|e^{-az} g_k^*(z, \eta)\|_{L_z^2} \left(\int_{\mathbb{R}} |\mathcal{F}_y(x_y(t, \cdot) v(t, z, \cdot))(\eta)|^2 dz \right)^{1/2}.$$

By (D.9) and Plancherel's theorem,

$$\begin{aligned} \|II_{k21}^3\|_{L^2[-\eta_0, \eta_0]} &\lesssim \left(\int_{-\eta_0}^{\eta_0} \int_{\mathbb{R}} |\mathcal{F}_y(x_y(t, \cdot) e^{az} v(t, z, \cdot))(\eta)|^2 dz d\eta \right)^{1/2} \\ &\lesssim \|x_y(t) v(t)\|_X. \end{aligned}$$

Since $\|x_y\|_{L^\infty} \lesssim \|x_y\|_Y^{1/2} \|x_{yy}\|_Y^{1/2} \lesssim \mathbb{M}_1(T) \langle t \rangle^{-1/2}$, we have

$$\|II_{k21}^3\|_{L^2[-\eta_0, \eta_0]} \lesssim \mathbb{M}_1(T) \mathbb{M}_2(T) \langle t \rangle^{-5/4},$$

By the Schwarz inequality,

$$\begin{aligned} \|II_{k22}^3\|_{L^\infty[-\eta_0, \eta_0]} &\lesssim \|v(t)\|_X \|x_y(t) \tilde{c}(t)\|_{L_y^2} \sup_{\eta \in [-\eta_0, \eta_0], c \in [2-\delta, 2+\delta]} \|e^{-az} g_{k3}^*(z, \eta, c)\|_{L_z^2} \\ &\lesssim \|v(t)\|_X \|\tilde{c}(t)\|_{L^\infty} \|x_y(t)\|_Y \lesssim \mathbb{M}_1(T)^2 \mathbb{M}_2(T) \langle t \rangle^{-3/2}. \end{aligned}$$

Combining the above, we have for $t \in [0, T]$,

$$\|R_k^5\|_Y \lesssim \|II_{k21}^3\|_{L^2[-\eta_0, \eta_0]} + \|II_{k22}^3\|_{L^\infty[-\eta_0, \eta_0]} \lesssim \mathbb{M}_1(T) \mathbb{M}_2(T) \langle t \rangle^{-5/4}.$$

Thus we complete the proof. \square

To estimate R_k^7 , we need the following.

Claim D.6. *There exist positive constants δ and C such that if $\sup_{t \in [0, T]} \|\tilde{c}(t)\|_Y \leq \delta$, then for $t \in [0, T]$,*

$$(D.10) \quad \|b - \tilde{c}\|_Y \leq C \|\tilde{c}\|_{L^\infty} \|\tilde{c}\|_Y, \quad \|b - \tilde{c}\|_{Y_1} \leq C \|\tilde{c}\|_Y^2,$$

$$(D.11) \quad \|b_y - c_y\|_Y \leq C \|\tilde{c}\|_{L^\infty} \|c_y\|_Y, \quad \|b_y - c_y\|_{Y_1} \leq C \|\tilde{c}\|_Y \|c_y\|_Y,$$

$$(D.12) \quad \|b_t - c_t\|_Y \leq C \|\tilde{c}\|_{L^\infty} \|c_t\|_Y,$$

$$(D.13) \quad \|b_{yy} - c_{yy}\|_Y \leq C (\|\tilde{c}\|_{L^\infty} \|c_{yy}\|_Y + \|c_y\|_{L^\infty} \|c_y\|_Y),$$

$$(D.14) \quad \|b_{yy} - c_{yy}\|_{Y_1} \leq C (\|\tilde{c}\|_Y \|c_{yy}\|_Y + \|c_y\|_Y^2),$$

$$(D.15) \quad \left\| \left(\frac{c}{2} \right)^{3/2} - 1 - \frac{3}{4} b \right\|_{L^2} \leq C \|\tilde{c}\|_{L^\infty} \|\tilde{c}\|_Y,$$

$$(D.16) \quad \left\| b - \tilde{c} - \frac{1}{8} \tilde{P}_1 \tilde{c}^2 \right\|_Y \leq C \|\tilde{c}\|_{L^\infty}^2 \|\tilde{c}\|_Y.$$

Proof. By (6.12),

$$b - \tilde{c} = \frac{4}{3} \tilde{P}_1 \left\{ \left(\frac{c}{2} \right)^{3/2} - 1 - \frac{3}{4} \tilde{c} \right\},$$

$$b_y - c_y = \tilde{P}_1 \{ (c/2)^{1/2} - 1 \} c_y, \quad b_t - c_t = \tilde{P}_1 \{ (c/2)^{1/2} - 1 \} c_t,$$

$$b_{yy} - c_{yy} = \tilde{P}_1 \{ (c/2)^{1/2} - 1 \} c_{yy} + \frac{1}{4} \tilde{P}_1 (c/2)^{-1/2} (c_y)^2.$$

Using the fact that $(c/2)^{3/2} - 1 - 3\tilde{c}/4 - 3\tilde{c}^2/32 = O(\tilde{c}^3)$, we can prove (D.10)–(D.14) and (D.16) in the same way as the proof of Claim B.6.

Finally, we will show (D.15). Let $\tilde{P}_2 = I - \tilde{P}_1$. Since $\tilde{P}_2 \tilde{c} = 0$ and

$$\left(\frac{c}{2}\right)^{3/2} - 1 - \frac{3b}{4} = \tilde{P}_2 \left\{ \left(\frac{c}{2}\right)^{3/2} - 1 \right\},$$

we have

$$\left\| \left(\frac{c}{2}\right)^{3/2} - 1 - \frac{3b}{4} \right\|_{L^2} = \left\| \tilde{P}_2 \left\{ \left(\frac{c}{2}\right)^{3/2} - 1 - \frac{3}{4} \tilde{c} \right\} \right\|_{L^2} \lesssim \|\tilde{c}\|_{L^4}^2.$$

Thus we complete the proof. \square

Claim D.7. *There exist positive constants δ and C such that if $\sup_{t \in [0, T]} \|\tilde{c}(t)\|_Y \leq \delta$, then for $t \in [0, T]$,*

$$(D.17) \quad \|\tilde{P}_1 R_1^7(s)\|_{Y_1} \leq C \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4}, \quad \|\tilde{P}_1 R_2^7(s)\|_{Y_1} \leq C \mathbb{M}_1(T)^2 \langle s \rangle^{-1},$$

$$(D.18) \quad \|\tilde{P}_1 R_1^7(s)\|_Y \leq C \mathbb{M}_1(T)^2 \langle s \rangle^{-3/2}, \quad \|\tilde{P}_1 R_2^7(s)\|_Y \leq C \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4},$$

$$(D.19) \quad \|\tilde{P}_1 \partial_y R_2^7(s)\|_{Y_1} \leq C \mathbb{M}_1(T)^2 \langle s \rangle^{-5/4}, \quad \|\tilde{P}_1 \partial_y R_2^7(s)\|_Y \leq C \mathbb{M}_1(T)^2 \langle s \rangle^{-3/2}.$$

Proof. In view of (6.14) and (4.2),

$$\begin{aligned} \|\tilde{P}_1 R_1^7\|_{Y_1} &\lesssim \|x_{yy}\|_Y \left\| \left(\frac{c}{2}\right)^{3/2} - 1 - \frac{3b}{4} \right\|_{L^2} \\ &\quad + \|b_{yy} - c_{yy}\|_{Y_1} + (\|b_y - c_y\|_Y + \|\tilde{c}\|_{L^\infty} \|c_y\|_Y) \|x_y\|_Y + \|c_y\|_Y^2, \end{aligned}$$

$$\|\tilde{P}_1 R_2^7\|_{Y_1} \lesssim \|\tilde{c}\|_Y \|x_{yy}\|_Y + \|c_y\|_Y \|x_y\|_Y + \|b_{yy} - c_{yy}\|_{Y_1} + \|c_y\|_Y^2.$$

Combining the above with Claim D.6, we have (D.17). We can obtain (D.18) and (D.19) in the same way. Thus we complete the proof. \square

APPENDIX E. LOCAL WELL-POSEDNESS IN EXPONENTIALLY WEIGHTED SPACE

The L^2 well-posedness of the KP-II equation around line solitons has been proved by Molinet, Saut and Tzvetkov ([29]) by using Bourgain's norm. In this section, we will explain well-posedness for exponentially localized initial data around a line soliton.

Let $u(t, x, y) = \varphi(x - 4t) + \tilde{v}(t, x - 4t, y)$ be a solution to (2.1). Then

$$(E.1) \quad \partial_t \tilde{v} = \mathcal{L} \tilde{v} - 3 \partial_x (\tilde{v}^2).$$

Proposition E.1. *Suppose $a > 0$ and $v_0 \in X \cap L^2(\mathbb{R}^2)$. If $\tilde{v}(0) = v_0$, then there exists a unique solution of (E.1) such that for any $T > 0$,*

$$(E.2) \quad \tilde{v} \in L^\infty(0, T; X) \cap X_T,$$

where X_T is the auxiliary Banach space used in Theorem 1.1 of [29]. If $v_0 \in H^1(\mathbb{R}^2)$ in addition, then $\tilde{v}(t) \in C([0, \infty); X)$.

Remark E.1. The Banach space X_T is continuously imbedded into $C([0, T]; L^2(\mathbb{R}^2))$. Moreover [29, Lemma 4.1] tells us that

$$(E.3) \quad \tilde{v}(t) \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^4(0, T; L^4(\mathbb{R}^2)),$$

and that $\tilde{v}(t) \in C([0, T]; H^s(\mathbb{R}^2))$ if $\tilde{v}(0) \in H^s(\mathbb{R}^2)$ for an $s \geq 0$.

Proof of Proposition E.1. To prove $\tilde{v}(t) \in L^\infty(0, T; X)$ for any $T > 0$, we will use the virial identity for the KP-II equation ([6]). Let $a > 0$, $\bar{p}(x) = 1 + \tanh ax$ and $p_n(x) = e^{2an}\bar{p}(x - n)$ for $n \in \mathbb{N}$. Suppose $v_0 \in H^3(\mathbb{R}^2) \cap X$ and $\partial_x^{-1}v_0 \in H^2(\mathbb{R}^2)$. Then by [29], we have $\tilde{v}(t) \in C([0, \infty); H^3)$ and $\partial_x^{-1}v_0 \in C([0, \infty); H^2)$. Multiplying (E.1) by $2p_n(x)v(t, x, y)$ and integrating the resulting equation over \mathbb{R}^2 , we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} p_n(x) \tilde{v}(t, x, y)^2 dx dy + \int_{\mathbb{R}^2} p'_n(x) \{3(\partial_x \tilde{v})^2 + 3(\partial_x^{-1} \partial_y \tilde{v})^2 - 4\tilde{v}^3\} dx dy \\ &= 3 \int_{\mathbb{R}^2} \{p'_n(x) \varphi(x) - p_n(x) \varphi'(x)\} \tilde{v}(t, x, y)^2 dx dy. \end{aligned}$$

By Claim 5.1 in [26],

$$\left| \int_{\mathbb{R}^2} \bar{p}'_n(x) \tilde{v}^3 dx dy \right| \leq C_1 \left(\int_{\mathbb{R}^2} \tilde{v}^2 dx dy \right)^{1/2} \left(\int_{\mathbb{R}^2} p'_n(x) \mathcal{E}(\tilde{v}) dx dy \right)^{1/2},$$

where C_1 is a constant independent of $n \in \mathbb{N}$. Hence there exists a positive constant C such that for every $n \in \mathbb{N}$,

$$\begin{aligned} & \int_{\mathbb{R}^2} p_n(x) \tilde{v}(t, x, y)^2 dx dy + 2 \int_0^t \int_{\mathbb{R}^2} \bar{p}'_n(x) \{(\partial_x \tilde{v})^2 + (\partial_x^{-1} \partial_y \tilde{v})^2\} (s, x, y) dx dy ds \\ & \leq \int_{\mathbb{R}^2} p_n(x) v_0(x, y)^2 dx dy + C \int_0^t \|\tilde{v}(s)\|_{L^2}^2 ds. \end{aligned}$$

By approximating a solution $\tilde{v}(t)$ of (E.1) with $\tilde{v}(0) = v_0 \in X \cap L^2(\mathbb{R}^2)$ by a sequence solutions $\{\tilde{v}_k(t)\}$ of (E.1) satisfying

$$\tilde{v}_k(0) \in H^3(\mathbb{R}^2), \quad \partial_x^{-1} \tilde{v}_k(0) \in H^2(\mathbb{R}^2), \quad \lim_{k \rightarrow \infty} \|\tilde{v}_k(0) - v_0\|_{L^2(\mathbb{R}^2)} = 0,$$

we have for any $v_0 \in L^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} p_n(x) \tilde{v}(t, x, y)^2 dx dy \leq \int_{\mathbb{R}^2} p_n(x) v_0(x, y)^2 dx dy + C \int_0^t \|\tilde{v}(s)\|_{L^2}^2 ds.$$

Passing to the limit $n \rightarrow \infty$, we obtain

$$(E.4) \quad \|\tilde{v}(t)\|_X^2 \leq \|v_0\|_X^2 + C \int_0^t \|\tilde{v}(s)\|_{L^2}^2 ds.$$

Since $\sup_{t \in [0, T]} \|\tilde{v}(t)\|_{L^2(\mathbb{R}^2)} < \infty$ for any $T > 0$, we have (E.2).

Suppose $v_0 \in H^1(\mathbb{R}^2) \cap X$. Then we have (E.2) and $\tilde{v}(t) \in C(\mathbb{R}; H^1(\mathbb{R}^2))$. By the variation of constants formula,

$$\tilde{v}(t) = e^{t\mathcal{L}_0} v_0 - 6 \int_0^t e^{(t-s)\mathcal{L}_0} \partial_x(\varphi \tilde{v}(s)) ds - 6 \int_0^t e^{(t-s)\mathcal{L}_0} \tilde{v}(s) \partial_x \tilde{v}(s) ds.$$

Since $\|e^{ax} \tilde{v}(s) \partial_x \tilde{v}(s)\|_{L^1} \leq \|\tilde{v}(s)\|_X \|\tilde{v}(s)\|_{H^1(\mathbb{R}^2)}$, we have $\tilde{v}(t) \in C([0, \infty); X)$ by using (3.6) and (3.10) in Lemma 3.4. Thus we complete the proof. \square

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